Demand Response With Capacity Constrained Supply Function Bidding

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***Abstract*—We study the problem faced by an operator who aims to allocate a certain amount of load adjustment (either load reduction or increment) to multiple consumers so as to minimize the aggregate consumer disutility. We propose and analyze a simple uniform-price market mechanism where every consumer submits a single bid to choose a supply function from a group of parameterized ones. These parameterized supply functions are designed to ensure that every consumer's load adjustment is within an exogenous capacity limit that is determined by the current power system operating condition. We show that the proposed mechanism yields bounded efficiency loss at a Nash equilibrium. In particular, the proposed mechanism is shown to achieve approximate social optimality at a Nash equilibrium, if the total capacity limit excluding the consumer with the largest one is much larger than the total amount of load to be adjusted. We complement our analysis through numerical case studies.**

***Index Terms*—Demand response, efficiency loss, game theory, Nash equilibrium, renewable generation.**

I. Introduction

THE electric power industry is undergoing a worldwide transition to a greater reliance on renewable energy resources so as to provide society with cleaner electricity supply. The variability of renewable generation imposes one of the most important challenges for the power system operation. The current approach for renewable energy integration is to balance the variability with dispatchable conventional generation. This approach works at today's modest penetration levels. In the near future, however, a huge amount of system reserve will be needed to balance the variability of the rapidly growing renewable generation [1]. Excess system reserve will induce considerable additional operating cost on the power system.

Clearly, there is an increasing need for flexible resources that can dynamically respond to the fast fluctuating renewable generation. An environmentally friendly option is to harness the flexibility in consumption on the demand side. Federal energy regulatory commission has identified the benefits of **coordination of demand-side resources** to balance the variability of intermittent renewable generation [2].

In this paper, we seek to design a proper market mechanism that coordinates the demand response of self-interested consumers. In particular, we consider a scenario where supply and demand are imbalanced, e.g., when there is a supply deficit or surplus due to the fluctuations in both renewable generation and system load. For example, an unexpected decrease in power generation (due to the shortfall in renewable generation or the failure of one conventional generating unit) may result in a rolling blackout. On the other hand, during a few hours of a day, renewable generation may be more than needed to meet the demand [3]. For example, in periods of minimum load with high amounts of renewable energy, the Hawaii Electric Light Company (HELCO) deploys down-reserves in addition to the curtailment of all wind generation [4]; California ISO (CAISO) is expected to have over-generation during the day by 2020 with even moderate wind [5]. In these cases, the system operator may call upon consumers to reduce or increase their consumption so as to match the supply [6].

A. Summary of Results

We study the problem faced by a system operator who aims to adjust a certain amount of electricity load at the minimum aggregate consumer utility loss. Since consumer cost functions (mappings from load adjustment to utility loss) are usually not available to the system operator, we seek to develop a market mechanism that incentivizes consumers to reveal their private information through properly designed monetary reward. We focus on uniform-price mechanisms under which all market participants have one-dimensional action spaces, and are faced with a single market-clearing price. Such mechanisms are simple to implement, and are usually considered to be fair among all market participants. The well-known Bertrand and Cournot competitions are examples of such market mechanisms. We note, however, that the Bertrand model is based on the assumption that a single market participant is willing and able to supply all demand[1] (which may not be satisfied in a demand response market since consumers cannot provide more load reduction than their

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[1]There are variants of the standard Bertrand model with capacity constraints on how much each market participant can supply. However, in such settings, pure Nash equilibrium may not exist [7].

current consumption), and that a Cournot oligopoly may result in unbounded efficiency loss at an equilibrium when the market demand is almost perfectly inelastic (see [8, Example 1]).

In the proposed market for demand-side management, every market participant (consumer) submits a single parameter to choose a particular supply function from a group of parameterized ones. Each submitted supply function specifies how much load adjustment the consumer would like to provide at a given price. Collecting the supply functions submitted from all consumers, a load service entity (LSE) or system operator sets a single uniform price to clear the market.

The set of linear supply functions serves as a natural candidate to implement parameterized supply function bidding. Each consumer submits the slope of her supply function, b, and provides bp amount of load adjustment at price p. We note that it is not straightforward to incorporate capacity constraints into linear supply functions (e.g., a consumer may be committed to supply more load reduction than her total consumption if the price is high). Further, under linear supply function bidding, arbitrary high efficiency loss is possible at a Nash equilibrium, e.g., when consumers have highly heterogeneous cost functions [9].

The proposed market therefore uses an alternative set of parameterized supply functions, which are designed to guarantee that every consumer will never provide more load adjustment than an exogenous capacity limit. Further, the particular form of supply functions considered in this paper enables us to derive upper bounds on the ratio of the aggregate cost resulting from a Nash equilibrium to the minimum aggregate cost. The established efficiency loss bound depends only on consumers' capacity limits and the total load adjustment, and holds for a set of general convex cost functions. For example, the proposed market is guaranteed to achieve approximate social optimality at a Nash equilibrium (i.e., the aggregate cost at a Nash equilibrium approximately equals the minimum aggregate cost), when there are only two consumers whose capacity limits are much larger than the total load adjustment, regardless of the cost functions of the two consumers (cf. the discussion following Theorem 5.1).

We show that the proposed uniform-pricing mechanism achieves social optimality at a competitive equilibrium. For an alternative setting with strategic consumers, we characterize the unique Nash equilibrium allocation as an optimal solution to a (strictly) convex optimization problem, and derive a nontrivial upper bound on the efficiency loss resulting from the unique Nash equilibrium allocation. The established upper bound is non-decreasing in the largest consumer's capacity constraint, and is strictly decreasing in the total capacity of other consumers. In particular, the proposed market mechanism achieves approximate social optimality at a Nash equilibrium, if the total capacity of all consumers (excluding the largest consumer) is much larger than the total amount of load adjustment. A qualitatively similar (upper) bound is also derived on the price markup above competitive levels at a Nash equilibrium.

We extend the proposed mechanism to a multistage setting where a certain amount of load needs to be adjusted in each period, and each consumer submits a single bid reflecting the consumer's willingness for demand adjustment over the entire demand response horizon. We focus on a setting where each consumer's disutility depends only on her total load adjustment over the entire horizon (e.g., when the consumer's load is deferrable). We derive equilibrium characterization and efficiency loss results that are qualitatively similar to those established for the single period model. In particular, approximate social optimality can still be achieved at a Nash equilibrium, if in every period, the total capacity of all consumers (excluding the largest consumer) is significantly larger than amount of load needed to adjusted.

B. Related Work

As an environmentally friendly approach to improve power system efficiency and to mitigate the variability of renewable generation, demand-side management has received extensive recent attention [10]–[12]. Various topics related to demand-side management include scheduling deferrable loads based on forecasted demand [13], designing energy market for price elastic demands [14], and quantifying the effect of demand response on wholesale electricity markets [15].

There exists a substantial literature on the design of incentive mechanisms for price-taking consumers, who do not anticipate the influence of their bids on market prices and therefore reveal their true economic and technical characteristics to the market maker [16]–[19]. It is noted that, in a demand response market where consumers provide load adjustment as product, some consumers may hold a large share of the market and can therefore influence the price [15], [20].

Game theoretic analysis has therefore been applied to study the behavior of selfish consumers who act strategically to maximize their own benefit. The authors of [21] study the design of efficient time-of-use (TOU) pricing schemes in the presence of strategic consumers. The existence and computation of a Nash equilibrium in a demand response market is studied in [22]. Closer to the present paper, a few recent works study the design of proper pricing schemes that achieve social optimality at a Nash equilibrium [23]–[25]. It is assumed in [23] and [24] that all players (electricity consumers) have the same preference on energy consumption, and as a result, the payoff of each player is simply the negative of her payment to the utility company. On the contrary, our model allows consumers to be heterogeneous by endowing each individual consumer with a cost function that maps load adjustment to her perceived disutility, and therefore incorporates the strategic tradeoff between payment and disutility faced by each individual consumer. The authors of [25] apply the Vickrey–Clarke–Groves (VCG) mechanism to demand-side management. Although VCG-type mechanisms are well known for being incentive compatible and efficient, VCG mechanism yields discriminatory prices to market participants. This is in sharp contrast to the uniform-price mechanism studied in this paper, under which all consumers always face a single market-clearing price. Further, under a VCG mechanism there may exist an equilibrium at which bidders use bids that they know will be rejected to punish its competitors.

This work is closely related to the literature on (unparametric) supply function equilibrium (SFE). In the seminal work [26], Klemperer and Meyer show that an enormous multiplicity of

SFEs may exist if there is no uncertainty in demand. This is because agents are allowed to submit arbitrary supply functions, and agents' complicated strategy space usually leads to the existence of multiple undesirable equilibria [27]. For the case with uncertainty in demand, on the other hand, a variety of conditions have been derived for the existence and uniqueness of SFE [26], [31], [28], [29], [30]. There is also a substantial literature that applies linear SFE models to analyze and characterize suppliers' bidding behavior in wholesale electricity markets [32]–[34]. More recently, the authors of [35] develop effective algorithms to compute an agent's best response (to her competitors' strategies) in a more general setting where agents are allowed to submit convex supply functions and transmission constraints are incorporated.

Closer to the present paper, efficiency loss of SFEs is analyzed numerically using empirical data from real-world electricity markets [36], [37]. The latter work in particular shows that the inefficiency of an oligopolistic wholesale electricity market usually increases with asymmetry in firms' cost functions. In this paper, we propose and analyze a parameterized supply function bidding mechanism, which yields bounded efficiency losses at the unique Nash equilibrium allocation of supply, in a general setting where market participants have asymmetric convex cost functions.

The remainder of this paper is organized as follows. In Section II, we introduce the system model and the market-clearing mechanism. In Section III, we show that aggregate consumer disutility is minimized at a competitive equilibrium. In Section IV, we show the existence of a Nash equilibrium and that all Nash equilibria must result in a unique allocation of load adjustment among consumers. We further characterize this equilibrium allocation as a solution to an optimization problem. This result enables us to derive an upper bound on efficiency loss of a Nash equilibrium in Section V. We extend our analysis to a multi-stage setting in Section VI. In Section VII, we present several numerical examples. Finally, in Section VIII, we make some concluding remarks and discuss future work.

II. Capacity-Constrained Supply Function Bidding

We consider a situation where there is a "deficit" or "surplus" in electricity supply, and an operator seeks to adjust (reduce or increase) G amount of system load. There are n consumers participating in the demand response program. Each consumer i can adjust at most D_i amount of electricity demand. To make the problem feasible and nontrivial, we assume that $G < \sum_{i=1}^n D_i$ and $n > 1$.

In our model, since the parameters G and $\{D_i\}_{i=1}^n$ are all positive, we let (without loss of generality) d_i be positive if consumer i provides a positive amount of desired load adjustment. In particular, when there is supply deficit (surplus), d_i is positive if consumer i reduces (increases, respectively) her energy consumption.

Each consumer i has a cost function that maps her load adjustment amount d_i to her disutility. We make the following assumption on each cost function C_i.

Assumption 2.1: For each consumer i, the cost function $C_i : (-\infty, \infty) \to (-\infty, \infty)$ is continuous and convex with $C_i(0) = 0$. $C_i()$ is strictly increasing over $[0, \infty)$. Over the domain $(-\infty, 0)$, $C_i()$ is either (i) nonnegative, or (ii) strictly increasing on $[-L_i, 0)$ and equals $C_i(-L_i)$ on $(-\infty, -L_i)$, where $L_i \geq 0$ is a constant. ■

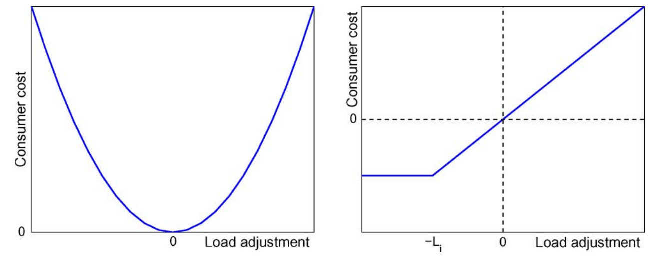

Fig. 1. Plot of two cost functions that satisfy Assumption 2.1.

The continuity of cost functions is a technical assumption that is usually made in the literature [27], [38], [17], [25]. The convexity of cost functions follows from the usual assumption that consumer utility function is concave, i.e., a consumer's marginal utility decreases with her energy consumption. While it is assumed in [38] that $C_i(d) = 0$ for every $d \leq 0$, our model is more general and incorporates the following two cases: 1) consumer i incurs cost for negative load adjustment, i.e., $C_i(d)$ is nonnegative over $(-\infty, 0]$, or 2) consumer i benefits (receives negative cost) from providing negative load adjustment, i.e., $C_i(d)$ is negative and strictly increasing over $[-L_i, 0]$. For the latter case, we have assumed that every consumer i's minimum cost is achieved at $d_i = -L_i$. Two example cost functions are presented in Fig. 1.

Ideally, if the operator could fully control consumers' demand response, she would solve the following optimization so as to minimize the aggregate cost:

$$\begin{aligned} \underset{(d_1,\ldots,d_n)}{\text{minimize}} \quad & \sum_{i=1}^n C_i(d_i) \\ \text{subject to} \quad & \sum_{i=1}^n d_i = G, \\ & -L_i \leq d_i \leq D_i, \qquad i = 1, \ldots, n, \qquad (1) \end{aligned}$$

where we have assumed, without loss of generality, that consumer i's load adjustment must lie in the interval $[-L_i, D_i]$.[2] We will refer to $\{d_i\}_{i=1}^n$ as a supply vector (or an allocation) and an optimal solution to (1) as a socially optimal allocation. The optimal allocation is hard to achieve in practice, and the minimum aggregate cost [the optimal value of problem (1)] will be only used as a benchmark.

Note that our model ignores the congestion in the distribution network. This is without loss of generality for the case with supply deficit, because load shedding will only alleviate the network congestion. For the case with supply surplus, we need to assume that the supply surplus is not severe enough to congest the power network.

We consider a market mechanism where each consumer submits a supply function to a load service entity (LSE) or a system operator; the submitted supply function describes the amount the electricity load she is willing to adjust as a function of the

[2]If consumer i's cost function is nonnegative over $(-\infty, 0]$, let $L_i = 0$.

market-clearing price. In particular, we restrict the set of supply functions that consumers can choose from to the following parameterized family:

$$d_i = S_i(b_i, p) = D_i - \frac{b_i}{p} \tag{2}$$

where $b_i \geq 0$ is the bid submitted by consumer i (a negative bid is not allowed in the mechanism), $S_i(b_i, p)$ denotes the amount of electricity demand consumer i would like to adjust at price $p > 0$, and $D_i \geq 0$ is the exogenous limit on consumer i's load adjustment. We use $\mathbf{b} = (b_1, \ldots, b_n)$ to denote the action vector of the n consumers.

The family of parameterized supply functions we consider in this paper is a generalization of the one studied in [38], where $D_i = G$ and $L_i = 0$ for every i. To study demand-side management for electric power systems, we generalize the model considered in [38] in the following two aspects.

1) In our model, each consumer has an exogenous limit on how much load adjustment she could provide. The capacity limit reflects physical constraints in power systems; for example, when there is deficit in supply, each consumer cannot provide more load reduction than her electricity consumption before load shedding.
2) A consumer can provide negative load adjustment (i.e., increase energy consumption when there is supply deficiency or decrease energy consumption when there is excessive supply). Our model makes no assumption on whether the consumer benefits from the negative load adjustment. This is an important feature because, for example, a consumer may either benefit or suffer from extra electricity consumption when load shedding is necessary.

As noted by [38], consumer i's bid b_i can be interpreted as the revenue that she is willing to forgo; or, from our perspective, the parameter b_i reflects consumer i's **unwillingness** to adjust a positive amount of demand. The supply function (2) implies that the load adjustment d_i increases if the capacity limit D_i increases, unwillingness b_i decreases, and the price p increases. All of these properties are intuitive, making (2) a valid model for demand response. We also note that the market mechanism allows consumers to adjust their demand in the "opposite" direction, i.e., $S_i(b_i, p)$ can be negative when $b_i > pD_i$. A consumer i may intend to supply a negative amount of goods if the cost function $C_i(d)$ is (sufficiently) negative when $d < 0$.

To clear the market, we have

$$\sum_{i=1}^{n} S_i(b_i, p) = \sum_{i=1}^{n} \left(D_i - \frac{b_i}{p} \right) = G. \tag{3}$$

Thus, the market clearing price is given by

$$p = \frac{\sum_{i=1}^{n} b_i}{-G + \sum_{i=1}^{n} D_i} \geq 0. \tag{4}$$

Note that when $\sum_{i=1}^{n} b_i = 0$ (i.e., $b_i = 0$ for every i) we have $p = 0$ and the supply function in (3) is undefined. To fix this, when $b_i = 0$ for every i, the load adjustment is allocated among consumers proportionally to their capacity limits, i.e.,

$$S_i(0, 0) = \frac{GD_i}{\sum_j D_j}.$$

Given a positive market-clearing price and the bid submitted by consumer i, the payoff of the consumer is

$$\begin{aligned}\pi_i(b_i, p) &= pS_i(b_i, p) - C_i(S_i(b_i, p)) \\ &= D_i p - b_i - C_i(S_i(b_i, p)).\end{aligned} \tag{5}$$

It is worth noting that, under the proposed mechanism, a consumer i who incurs cost for negative load adjustment[3] would never provide negative load adjustment at a competitive equilibrium or at a Nash equilibrium, because she could have obtained a strictly higher payoff by bidding pD_i and providing zero amount of load adjustment. For a price-taking consumer i, given the market price p she would like to bid pD_i to provide zero amount of load adjustment. For a strategic consumer i who can anticipate the impact of her bid on market price, given the bids submitted by other consumers $\{b_j\}_{j \neq i}$, she can bid

$$b_i = \frac{\sum_{j \neq i} b_j}{-G + \sum_{j \neq i} D_j} D_i$$

so as to provide zero load adjustment.[4] Therefore, consumer i's action at a (competitive or Nash) equilibrium remains unchanged, with the following modified cost function:

$$\tilde{C}_i(d_i) = \begin{cases} C_i(d_i), & \text{if } d_i \geq 0, \\ 0, & \text{if } d_i < 0. \end{cases}$$

For the purpose of analyzing consumer equilibrium behavior under the proposed mechanism, we can therefore (and will) focus only on case 2) where every consumer i's cost function is strictly increasing over $[0, -L_i]$ (cf. Assumption 2.1 for the definition of L_i), because the case with $C_i(d) \geq 0$ for $d \leq 0$ can be incorporated by letting $L_i = 0$.

III. Competitive Equilibrium

Here, we consider a setting where consumers act as price takers. Given a price p, a price-taking consumer i maximizes the payoff function in (5) over $b_i \geq 0$.

[3]The consumer has a cost function such that $C_i(d) \geq 0$ over $d \in (-\infty, 0]$; see the left-hand side of Fig. 1.

[4]If $b_j = 0$ for all $j \neq i$, then consumer i has to provide a positive amount of load even if she bids 0. In this case, consumer i may prefer to submit an arbitrarily small positive bid and supply a negative amount of load adjustment. Still, there does not exist a Nash equilibrium at which consumer i provides negative load adjustment, because she would always prefer to submit a smaller (positive) bid to lower the market price (that she has to pay).

Definition 3.1: A pair of action vector and price, $(\mathbf{b}, p)$, forms a competitive equilibrium if $p > 0$ and

$$\pi_i(b_i, p) = \max_{b_i' \geq 0} \pi_i(b_i', p), \quad i = 1, \ldots, n;$$
$$\sum_{i=1}^{n} S_i(b_i, p) = G.$$

Theorem 3.1 (Competitive Equilibrium): Suppose that Assumption 2.1 holds, $\sum_i D_i > G$, and that $n > 1$. There exists a competitive equilibrium. Further, any competitive equilibrium is an optimal solution to the optimization problem in (1), and is therefore socially optimal. ■

The proof of Theorem 3.1 is deferred to Appendix A. This theorem shows that the proposed mechanism achieves social optimality (i.e., minimizes the total consumer disutility resulting from load adjustment), if all consumers are price-taking.

IV. Nash Equilibrium: Existence and Characterization

Here, we consider consumers with market power who bid strategically to maximize their own payoff. We will use the game-theoretic Nash equilibrium concept to study consumer behavior. We write consumer i's payoff as a function of the action vector given in (6), shown at the bottom of the page, where $\mathbf{b}_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. We note that, when $\sum_{j \neq i} b_j = 0$, $\pi_i(b_i, \mathbf{b}_{-i})$ is discontinuous at $b_i = 0$.

We will first derive a necessary condition for the existence of a Nash equilibrium in Lemma 4.1, and then characterize the unique supply vector (load adjustment allocation) resulting from any Nash equilibrium as a solution to an optimization problem in Theorem 4.1 (the main result of this section).

Definition 4.1: A vector $\mathbf{b}$ is a (pure-strategy) Nash equilibrium if

$$\pi_i(b_i, \mathbf{b}_{-i}) = \max_{b_i' \geq 0} \pi_i(b_i', \mathbf{b}_{-i}), \qquad i = 1, \ldots, n.$$

In this paper, we focus on pure-strategy Nash equilibria, which will be referred to as Nash equilibria in the rest of this paper. Before introducing the main result of this section (existence and characterization of Nash equilibria), we provide some useful mathematical preliminaries on Nash equilibria. The following lemma shows the nonexistence of Nash equilibria in the "pivotal" case where the inelastic demand adjustment cannot be met by the $n-1$ "smallest" consumers.

Lemma 4.1: Suppose that Assumption 2.1 holds and that $\sum_{j \neq i} D_j < G$ for some i. There does not exist a Nash equilibrium.

Proof: Substituting (4) to (2), we write the supply provided by consumer i as a function of the bidding vector $\mathbf{b}$ as follows:

$$d_i = D_i - \frac{b_i(-G + \sum_j D_j)}{\sum_j b_j}.$$

We note that d_i decreases and converges to $G + D_i - \sum_j D_j > 0$, as b_i increases to infinity. On the other hand, the market clearing price p blows up as b_i grows large. Therefore, consumer i's payoff is unbounded as her bid b_i increases to infinity. A Nash equilibrium does not exist. ■

Lemma 4.1 shows that in the "pivotal" case, the proposed market fails since one consumer would have the monopoly power. We note that a properly chosen price cap (i.e., a limit on the maximum market-clearing price possibly set by the market maker) can ensure the existence of a Nash equilibrium, even if the conditions required by Lemma 4.1 hold.

The proposed market may also fail if $\sum_{j \neq i} D_j = G$ for some i. In this case, it can be shown that there exist infinitely many Nash equilibria; all of these equilibria are of the same form: for every $j \neq i$, $b_j = 0$, and $b_i > 0$ is large enough so that the market price is higher than every consumer j's marginal cost at D_j. The allocation resulting from these Nash equilibria is unique: consumer i supplies zero and other consumers' supplies equal their capacity limits. At such a Nash equilibrium, consumer i provides no load adjustment but can determine the market price. As a result, arbitrary high market price is supported at an equilibrium, and arbitrary high efficiency loss is possible, for example, when consumer i has the lowest cost among all consumers.

Lemma 4.1 and the discussion above motivate the following assumption, which guarantees that at every Nash equilibria, the price is determined by at least two market participants. This nice property enables us to bound the efficiency loss at a Nash equilibrium in Section V.

$$\pi_i(b_i, \mathbf{b}_{-i}) = \begin{cases} \dfrac{D_i \sum_{j=1}^{n} b_j}{-G + \sum_{j=1}^{n} D_j} - b_i - C_i\left(D_i - \dfrac{b_i(-G + \sum_{j=1}^{n} D_j)}{\sum_{j=1}^{n} b_j}\right), & \text{if} \sum_{j=1}^{n} b_j > 0 \\ -C_i\left(\dfrac{G D_i}{\sum_j D_j}\right), & \text{if} \sum_{j=1}^{n} b_j = 0 \end{cases} \tag{6}$$

Assumption 4.1: For every i, $\sum_{j\neq i}^{n} D_j > G$. ■

Lemma 4.2: Suppose that Assumptions 2.1 and 4.1 hold. If $\mathbf{b}$ is a Nash equilibrium, then it has at least two positive components.

Proof: Suppose that $\mathbf{b}$ is a Nash equilibrium such that $b_j = 0$ for every $j \neq i$. Consumer i's payoff is given by

$$\pi_i(b_i, \mathbf{0}) = \begin{cases} \dfrac{D_i b_i}{-G + \sum_{j=1}^{n} D_j} - b_i \\ -C_i\left(D_i + G - \sum_{j=1}^{n} D_j\right), & \text{if } b_i > 0 \\ -C_i\left(\dfrac{GD_i}{\sum_j D_j}\right), & \text{if } b_i = 0. \end{cases}$$

According to Assumption 4.1, we have $\sum_j D_j - G > D_i$. It follows that consumer i would like to submit an arbitrarily low (but positive) bid to minimize the market clearing price. The vector $\mathbf{b}$ cannot be a Nash equilibrium. ■

We introduce some notations that will be useful for the rest of the paper. Under a given action vector $\mathbf{b}$, the load adjustment provided by consumer i is

$$S_i(b_i, p(\mathbf{b})) = D_i - \frac{b_i}{p(\mathbf{b})} = D_i - \frac{b_i(-G + \sum_{i=1}^{n} D_i)}{\sum_{i=1}^{n} b_i}$$

where $p(\mathbf{b})$ is the market clearing price given in (4).

Theorem 4.1 (Nash Equilibrium): Suppose that Assumptions 2.1 and 4.1 hold. We have the following.

1) There exists a Nash equilibrium. Further, there exists a unique Nash equilibrium if every consumer's cost function is continuously differentiable.
2) For any Nash equilibrium $\mathbf{b}$, the resulting allocation $\{S_i(b_i, p(\mathbf{b}))\}_{i=1}^{n}$ is the unique solution to the following optimization problem:

$$\begin{aligned} \min_{(d_1,\ldots,d_n)} \quad & \sum_{i=1}^{n} \hat{C}_i(d_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} d_i = G. \\ & -L_i \leq d_i \leq D_i, \qquad i = 1, \ldots, n \end{aligned} \tag{7}$$

where the function $\hat{C}_i(d_i)$ is defined in (8), shown at the bottom of the page. ■

The proof of Theorem 4.1 is deferred to Appendix B. There may exist multiple Nash equilibria when consumers' marginal cost functions are discontinuous; we note, however, that all Nash equilibria must lead to the unique allocation of load adjustment that solves problem (7). The following proposition shows that individual rationality is guaranteed at a Nash equilibrium, and is proved in Appendix C.

Proposition 4.1 (Individual Rationality): Suppose that Assumptions 2.1 and 4.1 hold. At a Nash equilibrium, every consumer achieves a nonnegative payoff, and each consumer with a non-zero supply obtains a positive payoff. ■

Before ending this section, we discuss how capacity limits affect the equilibrium allocation. Presumably an increase in one consumer's capacity limit would raise her supply. However, the following proposition shows the opposite.

Proposition 4.2: Let $\{d_j\}_{j=1}^{n}$ and $\{\bar{d}_j\}_{j=1}^{n}$ denote the supply vectors resulting from two Nash equilibria under the capacity limit vectors $\{D_j\}_{j=1}^{n}$ and $\{\bar{D}_j\}_{j=1}^{n}$, respectively. Suppose that Assumptions 2.1 and 4.1 hold, and that $\bar{D}_j = D_j$ for all $j \neq i$. If $d_i < D_i$ and $\bar{D}_i > D_i$, then $d_i \geq \bar{d}_i$. ■

The proof of Proposition 4.2 is given in Appendix D. We observe from (8) that an increase of consumer i's capacity keeps his or her modified cost function unchanged while reducing the modified cost of the other consumers. As a result, at a Nash equilibrium where the aggregate modified cost is minimized (cf. Theorem 4.1), consumers other than i tend to increase their supply, which will in turn decrease the supply of consumer i.

$$\hat{C}_i(d_i) = \begin{cases} \left(1 + \dfrac{d_i}{-G + \sum_{j\neq i} D_j}\right) C_i(d_i) - \dfrac{1}{-G + \sum_{j\neq i} D_j} \displaystyle\int_0^{d_i} C_i(x)dx, & \text{if } d_i \geq 0 \\ \left(1 + \dfrac{d_i}{-G + \sum_{j\neq i} D_j}\right) C_i(d_i) + \dfrac{1}{-G + \sum_{j\neq i} D_j} \displaystyle\int_{d_i}^{0} C_i(x)dx, & \text{if } d_i < 0 \end{cases} \tag{8}$$

V. Bounds on Efficiency Loss and Market Price

Here, we derive an upper bound on the efficiency loss at a Nash equilibrium. To measure the price markup (above competitive levels) in the proposed market, we also establish an upper bound on the *Lerner index* of consumers who supply less than their capacity limits.

We first note that arbitrarily high efficiency loss is possible if it is socially optimal to allocate a negative amount of load adjustment to some consumer. This observation motivates the following assumption.

Assumption 5.1: There exists a socially optimal allocation of which every component is nonnegative. That is, there exists some $\mathbf{d}^*$ that is an optimal solution to (1) such that $d_i^* \geq 0$ for every i. ■

This assumption requires that every consumer i's marginal cost at zero supply, $\partial^- C_i(0)$,[5] is no more than the marginal cost of any consumer who provides a positive amount of load adjustment at the social optimum. It is guaranteed to hold if $\partial^- C_i(0) \leq \partial^+ C_j(0)$ for every pair of consumers i and j.

Theorem 5.1 (Bounded Efficiency Loss): Suppose that Assumptions 2.1, 4.1, and 5.1 hold, and that i denotes the consumer who has the largest capacity limit, i.e., $i \in \arg\max_j \{D_j\}$. Let $\mathbf{d}^*$ be a socially optimal allocation (an optimal solution to (1)), and $\mathbf{d}$ be an allocation resulting from a Nash equilibrium, respectively. We have

$$\sum_{j=1}^{n} C_j(d_j) \leq \left(1 + \frac{\min\{D_i, G\}}{-G + \sum_{j \neq i} D_j}\right) \sum_{j=1}^{n} C_j(d_j^*).$$

Further, the bound is tight for the case with $D_i \geq G$: for any $\epsilon > 0$, $n \geq 2$, and $D_i \geq G > 0$, there exists a choice of $\{D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n\}$ and cost functions $\{C_j\}_{j=1}^n$ such that $D_i = \max_j\{D_j\}$ and

$$\sum_{j=1}^{n} C_j(d_j) \geq \epsilon + \left(1 + \frac{G}{-G + \sum_{j \neq i} D_j}\right) \sum_{j=1}^{n} C_j(d_j^*).$$

■

The proof of Theorem 5.1 is given in Appendix E. We note that the derived bound depends only on the capacity limits $(D_1, \ldots, D_n)$ and total supply deficit G, regardless of the cost functions $(C_1, \ldots, C_n)$. For a special case where $L_i = 0$ and $D_i = G$ for every i, Assumptions 4.1 and 5.1 naturally hold, and Theorem 5.1 degenerates to [38, Theorem 2].

The incorporation of consumers' (heterogeneous) capacity limits enables us to derive insights on how the size of market participants influences the efficiency loss of the proposed market mechanism. Given a fixed demand (for load adjustment) G, it is straightforward to check that the efficiency loss upper bound derived in Theorem 5.1 is increasing with the capacity limit of the largest consumer D_i, and is decreasing in the total capacity of the other consumers $\sum_{j \neq i} D_j$. Indeed, as $-G + \sum_{j \neq i} D_j$ converges to zero, it is possible that a Nash equilibrium yields arbitrarily high efficiency loss (cf. the example constructed in the proof of Theorem 5.1). On the other hand, the market mechanism considered in this paper guarantees approximate social optimality at all Nash equilibria, if there exist at least two consumers with large capacity limits, or a large number of (small) consumers. For example, if there are m ($m \leq n$) consumers each with a capacity limit larger than $\epsilon G > 0$, then the ratio of the aggregate cost resulting from a Nash equilibrium and that resulting from a socially optimal allocation is upper bounded by (cf. Theorem 5.1)

$$1 + \frac{1}{\epsilon(m-1) - 1}$$

which converges to 1 as m grows large.

Before ending this section, we derive an upper bound on the Lerner index, which is commonly used to measure a market participant's market power. Given a strategy profile $\mathbf{b}$, we define the Lerner index of consumer i as (in [39])

$$\text{Lerner}_i(\mathbf{b}) = \frac{p(\mathbf{b}) - \dfrac{\partial^+ C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i}}{p(\mathbf{b})}$$

where $p(\mathbf{b})$ is the market-clearing price resulting from the bid vector $\mathbf{b}$, and $\partial^+ C_i(S_i(b_i, p(\mathbf{b})))$ is the right directional derivative of consumer i's cost at her load adjustment level $S_i(b_i, p(\mathbf{b}))$. The Lerner index ranges in [0, 1], and a higher Lerner index is an indicator of higher market power.

Corollary 5.1: Suppose that Assumptions 2.1 and 4.1 hold. At a Nash equilibrium $\mathbf{b}$, if consumer i's supply is strictly less than D_i, then we have

$$\text{Lerner}_i(\mathbf{b}) \leq \frac{D_i}{-G + \sum_j D_j}.$$

Proof: Since $S_i(b_i, p(\mathbf{b})) < D_i$, we have $b_i > 0$. We have shown in the proof of Theorem 4.1 (cf. Appendix B) that

$$\begin{aligned}\frac{\partial^+ C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i} &\geq \frac{\partial^+ C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i}\left(1 - \frac{b_i}{\sum_j b_j}\right) \\ &\geq \frac{-G + \sum_{j \neq i} D_j}{-G + \sum_j D_j} p(\mathbf{b})\end{aligned}$$

where the second inequality follows from (22). The preceding inequality yields the bound claimed in the corollary, after rearranging terms. ■

We note that even when all consumers have high cost and provide small load adjustment, the market-clearing price cannot be too high because every consumer's Lerner index is bounded. Lemma 4.2 implies that at a Nash equilibrium, at least two consumers' load adjustments are strictly less than their capacity limits. Therefore, the bound derived in Corollary 5.1 applies

[5]Note that we have not assumed that C_i is differentiable. We use the notation $\partial^+ C_i$ and $\partial^- C_i$ to denote the right and left directional derivative of C_i, respectively.

to at least two consumers. Let consumer i be one of the consumers whose supply is less than her capacity limit. As the total capacity limit of all consumers (excluding consumer i) grows much larger than the total amount of load adjustment G, the bound derived in Corollary 5.1 converges to zero, which implies that the marginal cost of consumer i approximately equals the market-clearing price.

VI. Extension to Multistage Markets

Here, we generalize the model formulated in Section II by considering a multistage setting. We study a simple mechanism under which each consumer submits a single bid that reflects her willingness to adjust her demand over the entire T stages. Such mechanisms are easy to implement, and require the minimum effort from participating consumers. We will establish results on equilibrium characterization and bounded efficiency loss that are analogous to those derived in Sections IV and V.

Suppose that the system operator aims to adjust G_t amount of load at stage t, for $t = 1, \ldots, T$. Ideally, with full information on consumers' preferences, the system operator would like to solve the following optimization so as to minimize the aggregate cost (maximize the aggregate welfare):

$$\begin{aligned} \underset{(\mathbf{d}_1,\ldots,\mathbf{d}_n)}{\text{minimize}} \quad & \sum_{i=1}^{n} C_i\left(\sum_t d_{i,t}\right) \\ \text{subject to} \quad & \sum_{i=1}^{n} d_{i,t} = G_t, \qquad t = 1,\ldots,T, \\ & -L_{i,t} \le d_{i,t} \le D_{i,t}, \qquad \forall i, \quad \forall t \end{aligned} \tag{9}$$

where $d_{i,t}$ is the load adjustment provided by consumer i at stage t, $\mathbf{d}_i = (d_{i,1}, \ldots, d_{i,T})$, and $D_{i,t} > 0$ is the maximum amount of load consumer i could adjust at stage t. By some slight abuse of notation, we will let

$$L_i = \sum_{t=1}^{T} L_{i,t}, \quad D_i = \sum_{t=1}^{T} D_{i,t}, \quad d_i = \sum_{t=1}^{T} d_{i,t}. \tag{10}$$

Note that we have assumed that every consumer i's disutility (resulting from demand response) depends only on its total load adjustment, d_i. This assumption holds for deferrable load such as plug-in hybrid electric vehicles (PHEVs) and dish washers, which requires a certain amount of energy consumption before a pre-determined deadline. Indeed, our setting incorporates the demand response of n deferrable tasks with deadlines no earlier than the final stage T. Each consumer i's cost function can be given by

$$C_i(d_i) = \begin{cases} q_i d_i, & \text{if } d_i \le D_i', \\ q_i D_i' + H_i(d_i - D_i'), & \text{otherwise} \end{cases}$$

where D_i' is the maximum amount of load reduction that the consumer can provide over the T stages such that her task can still be completed by its deadline. Here, $q_i > 0$ denotes consumer i's marginal disutility resulting from deferring the completion of her task, and $H_i()$ reflects the consumer's disutility resulting from not completing the task before its deadline. It is straightforward to check that $C_i()$ satisfies Assumption 2.1, as long as $H_i()$ is strictly increasing and convex with $H_i(0) = 0$ and $H_i'(0) \ge q_i$.

Every consumer i submits a bid b_i to the system; her supply function at stage t is determined by her capacity limit $D_{i,t}$

$$d_{i,t} = S_{i,t}(b_i, p_t) = D_{i,t} - \frac{b_i}{p_t}. \tag{11}$$

Note that consumers are allowed to have time-variant capacity limits. To clear the market, we have

$$\sum_{i=1}^{n} S_{i,t}(b_i, p_t) = \sum_{i=1}^{n}\left(D_{i,t} - \frac{b_i}{p_t}\right) = G_t$$

which implies that

$$p_t = \frac{\sum_{i=1}^{n} b_i}{-G_t + \sum_{i=1}^{n} D_{i,t}} \ge 0, \qquad t = 1, \ldots, T. \tag{12}$$

We let $S_{i,t}(0,0) = G_t/n$ for every i and every t, if $\sum_{i=1}^{n} b_i = 0$. Given a sequence of market-clearing prices and the bid submitted by consumer i, her payoff is

$$\pi_i(b_i, \mathbf{p}) = \sum_{t=1}^{T} p_t S_{i,t}(b_i, p_t) - C_i\left(\sum_{t=1}^{T} S_{i,t}(b_i, p_t)\right)$$

where $\mathbf{p} = (p_1, \ldots, p_T)$. A game is naturally defined by writing every consumer i's payoff as a function of the bids submitted by all consumers in (13), shown at the bottom of the following page.

The following assumption is analogous to Assumption 4.1.

Assumption 6.1: We assume that $\sum_{j\neq i}^{n} D_{j,t} > G_t$, for every i and every t. ■

Under Assumptions 2.1 and 6.1, it is easy to check that Lemma 4.2 holds, i.e., a Nash equilibrium $\mathbf{b} = (b_1, \ldots, b_n)$ must have at least two positive components. We are now ready to introduce the main result of this section, in the following two theorems.

Theorem 6.1 (Nash Equilibrium): Suppose that Assumptions 2.1 and 6.1 hold. There exists a Nash equilibrium. Further, the allocation resulting from any Nash equilibrium, $(d_1, \ldots, d_n)$, is the unique solution to the following optimization problem:

$$\begin{aligned} \underset{(d_1,\ldots,d_n)}{\text{minimize}} \quad & \sum_{i=1}^{n} \hat{C}_i(d_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} d_i = \sum_{t=1}^{T} G_t, \\ & -L_i \le d_i \le D_i, \qquad i = 1, \ldots, n \end{aligned} \tag{14}$$

where the function $\hat{C}_i(d_i)$ is defined in (15), shown at the bottom of the following page, with $R_t \triangleq -G_t + \sum_j D_{j,t}$. ■

Before introducing the efficiency lower bound for Nash equilibria, we make the following assumption that is analogous to Assumption 5.1.

Assumption 6.2: There exists a socially optimal allocation at which every consumer provides nonnegative total load adjustment. That is, there exists an optimal solution to (9) such that the total supply provided by every consumer i, d_i^* [see its definition in (10)], is nonnegative. ■

Theorem 6.2 (Bounded Efficiency Loss): Suppose that Assumptions 2.1, 6.1, and 6.2 hold. Let $\mathbf{d}^*$ be an optimal solution to (9), and $\mathbf{d}$ be an allocation resulting from a Nash equilibrium, respectively. We have

$$\sum_{j=1}^{n} C_j(d_j) \leq \frac{1}{1-\sum\limits_{t}\frac{D_{i^t,t}}{(TR_t)}}\sum_{j=1}^{n} C_j(d_j^*)$$

where $i^t = \arg\max_j\{D_{j,t}\}$ is a consumer with the largest capacity at stage t, and $R_t = -G_t + \sum_j D_{j,t}$. ■

The proof of Theorem 6.2 is given in Appendix G. Theorem 6.2 implies that every Nash equilibrium is approximately socially optimal, if the following ratio:

$$\frac{D_{i^t,t}}{TR_t} = \frac{D_{i^t,t}}{T(-G_t+\sum\limits_{j} D_{j,t})},$$

is close to zero for every t. This must be the case, if at every stage t, the total capacity limit of all consumers (excluding the largest consumer i^t), $\sum_{j\neq i^t} D_{j,t}$, is much larger than the total amount of load adjustment G_t. This condition that suffices approximate social optimality is qualitatively similar to the insight behind Theorem 5.1.

VII. Numerical Results

Here, we present numerical examples to demonstrate the performance of the proposed market mechanism. For simplicity, we let $T=1$. Suppose that consumer 1 has the largest capacity limit $D_1 \geq 10$. The capacity limit of consumer 2 is 10, and $D_i = 1$ for $i \geq 3$. The total load adjustment $G = 10$. There are m small consumers with capacity limit 1 (there are $m+2$ consumers in total). All consumers have linear costs. In the following two subsections, we consider two different scenarios where the largest consumer has the lowest, or the highest marginal cost among all consumers.

A. Largest Consumer Has the Lowest Marginal Cost

We first consider the case where the largest consumer has the lowest marginal cost. Suppose that consumer 1's marginal cost is 1, while all of the other consumers' marginal cost is 2. In this case, it is socially optimal to reduce only the energy consumption of consumer 1, i.e., $d_1 = 10$, and $d_i = 0$ for all $i \geq 2$, It is straightforward to see that the minimum aggregate cost is 10, regardless of the number of small consumers.

For different values of m and D_1, we compute the unique allocation resulting from Nash equilibria through the optimization problem in (7). We list the calculation results in Table I, where d_i is the amount of load adjustment provided by consumer i at a Nash equilibrium, p is the equilibrium market price, and Lerner$_i$ is the Lerner index of consumer i. We note that the Lerner index is the same for all consumers $i \geq 2$ since they have the same marginal cost. We observe from Table I that, as the number of small consumers increases, the most efficient consumer 1 usually provides more load adjustment; as a result, the equilibrium allocation usually becomes more efficient as m increases, as

$$\pi_i(b_i,\mathbf{b}_{-i}) = \begin{cases} -b_iT + \sum\limits_{t=1}^{T}\dfrac{D_{i,t}\sum\limits_{j=1}^{n} b_j}{-G_t+\sum\limits_{j=1}^{n} D_{j,t}} - C_i\left(\sum\limits_{t=1}^{T}\left(D_{i,t} - \dfrac{b_i(-G_t+\sum\limits_{j=1}^{n} D_{j,t})}{\sum\limits_{j=1}^{n} b_j}\right)\right), & \text{if} \sum\limits_{j=1}^{n} b_j > 0, \\ -C_i\left(\sum\limits_{t=1}^{T}\left(\dfrac{G_tD_{i,t}}{\sum\limits_{j} D_{j,t}}\right)\right), & \text{if} \sum\limits_{j=1}^{n} b_j = 0 \end{cases} \tag{13}$$

$$\hat{C}_i(d_i) = \begin{cases} \dfrac{1}{T-\sum\limits_{t}\left(\dfrac{D_{i,t}}{R_t}\right)}\left[\left(1-\dfrac{\sum\limits_{t} D_{i,t} - d_i}{\sum\limits_{t} R_t}\right)C_i(d_i) - \dfrac{1}{\sum\limits_{t} R_t}\displaystyle\int_0^{d_i} C_i(x)dx\right], & \text{if } d_i \geq 0, \\ \dfrac{1}{T-\sum\limits_{t}\left(\dfrac{D_{i,t}}{R_t}\right)}\left[\left(1-\dfrac{\sum\limits_{t} D_{i,t} - d_i}{\sum\limits_{t} R_t}\right)C_i(d_i) + \dfrac{1}{\sum\limits_{t} R_t}\displaystyle\int_{d_i}^{0} C_i(x)dx\right], & \text{if } d_i < 0 \end{cases} \tag{15}$$

TABLE I
UNIQUE EQUILIBRIUM ALLOCATION WITH $C_1(d) = d$, $D_1 = 10$, AND $D_1 = 20$

Number of small consumers m	2	4	6	8	10
$d_1 : (D_1 = 10, D_1 = 20)$	6, 3.5	5.333, 4.727	6.444, 6.27	8.2, 8.128	10, 10
$d_2 : (D_1 = 10, D_1 = 20)$	2, 4.5	0.6667, 1.273	0.222, 0.36	0.1, 0.144	0, 0
$d_3 : (D_1 = 10, D_1 = 20)$	1, 1	1, 1	0.556 , 0.562	0.213, 0.216	0, 0
$p : (D_1 = 10, D_1 = 20)$	4, 2.75	2.333, 2.182	2.074, 2.045	2.025, 2.016	2, 2
$\text{Lerner}_1 : (D_1 = 10, D_1 = 20)$	0.75, 0.636	0.571, 0.542	0.518 , 0.511	0.506, 0.504	0.5, 0.5
$\text{Lerner}_2 : (D_1 = 10, D_1 = 20)$	0.5, 0.273	0.143, 0.083	0.0357, 0.022	0.123, 0.008	0, 0

TABLE II
UNIQUE EQUILIBRIUM ALLOCATION WITH $C_1(d) = 2.5d$, $D_1 = 10$, AND $D_1 = 20$

Number of small consumers m	2	4	6	8	10
$d_1 : (D_1 = 10, D_1 = 20)$	3.333, 0.588	2.222, 0.465	1.111, 0	0, 0	0, 0
$d_2 : (D_1 = 10, D_1 = 20)$	4.667, 7.412	3.778, 5.535	2.889, 4	2, 2	0.5, 0.645
$d_3 : (D_1 = 10, D_1 = 20)$	1, 1	1, 1	1, 1	1, 1	0.95, 0.936
$p : (D_1 = 10, D_1 = 20)$	6.667, 3.235	3.889, 2.791	2.963, 2.5	2.5, 2.222	2.1, 2.0645
$\text{Lerner}_1 : (D_1 = 10, D_1 = 20)$	0.625, 0.227	0.357, 0.104	0.156, 0	0, -0.125	-0.19, -0.211
$\text{Lerner}_2 : (D_1 = 10, D_1 = 20)$	0.7, 0.382	0.486, 0.283	0.325, 0.2	0.2, 0.1	0.048, 0.031

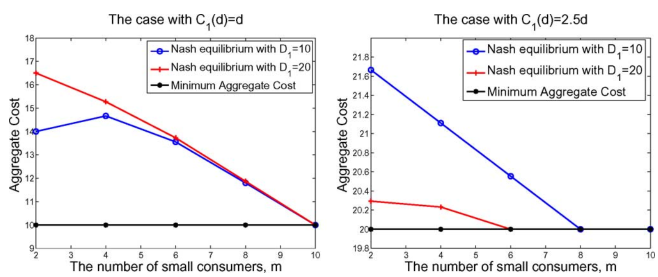

Fig. 2. Plot of aggregate cost at the unique Nash equilibrium and the minimum aggregate cost (resulting from a socially optimal allocation), with different values of D_1 and m.

shown in the left subplot of Fig. 2. The only exception happens when $m = 4$, because for both cases with $m = 2$ and $m = 4$, the supply provided by inefficient small consumers reach their capacity limits, and when $m = 2$ the most efficient consumer 1 is able to provide more load adjustment at a Nash equilibrium. We also observe from the left figure of Fig. 2 that social optimality is achieved at a Nash equilibrium with 10 small consumers.

We observe from Table I that the market price as well as the Lerner indexes (of all consumers) decrease with m, i.e., a larger number of small consumers introduces more competition and therefore reduces the market price as well as the Lerner indexes. It can also be seen from Table I that a higher D_1 leads to lower market prices; this is intuitive since the market price is determined by (4). We note, however, from the left subplot of Fig. 2 that a higher D_1 always leaFds to a higher aggregate cost at a Nash equilibrium. This is because a higher D_1 always results in less supply from the most efficient consumer 1 (cf. Proposition 4.2). We finally note that the numerical results verify the results derived in Theorem 5.1: the ratio of the aggregate cost at an equilibrium to the minimum possible aggregate cost never exceeds the upper bound established in Theorem 5.1.

B. Largest Consumer Has the Highest Marginal Cost

Here, we consider an alternative scenario where the largest consumer has the highest marginal cost. We set the marginal cost of consumer 1 to be 2.5. All the other parameters remain the same as in Section VII-A. It is easy to check that the minimum aggregate cost is 20, which can be achieved if and only if consumer 1 provides no load adjustment.

For different values of m and D_1, the unique allocation at a Nash equilibrium is computed through the optimization problem in (7). Proposition 4.2 shows that the load adjustment provided by a consumer decreases with her capacity (if all the other parameters keep unchanged). In our setting, the most inefficient consumer has the largest capacity, which prevents her from providing load adjustment at a Nash equilibrium. As a result, comparing the two subplots in Fig. 2 we conclude that the efficiency loss at an equilibrium with $C_1(d) = 2.5d$ (depicted on the right subplot of Fig. 2) is much smaller than that with $C_1(d) = d$.

We observe from Table II that the market price and Lerner indexes decrease with the number of (efficient) small consumers m. Different from the results in the left subplot of Fig. 2, we observe from the right subplot of Fig. 2 that an increase in D_1 reduces the aggregate cost when $m = 2$. This is because a higher D_1 reduces the supply (of load adjustment) from the most inefficient consumer 1 (see Proposition 4.2). Again, it can be seen from Table II that a higher D_1 leads to lower market prices and lower Lerner indexes. We note that when $m \geq 8$ the Lerner index of consumer 1 could be negative, because the market price goes below the consumer's marginal cost.

C. Case With Uncertain Load Adjustment

Here, we test the performance of the proposed mechanism in a setting where the actual demand adjustment G is uncertain and the consumers decide their bids on the basis of the expected value of G. We fix $D_1 = 10$, and consider the two different cases with $C_1(d) = d$ (the setting of Section VII-A) and with $C_1(d) = 2.5d$ (the setting of Section VII-B). For each of these two cases, we compute the expected aggregate cost for two distributions of G: a uniform distribution over [8, 12] and another uniform distribution over [5, 15]. We note that for all cases, the

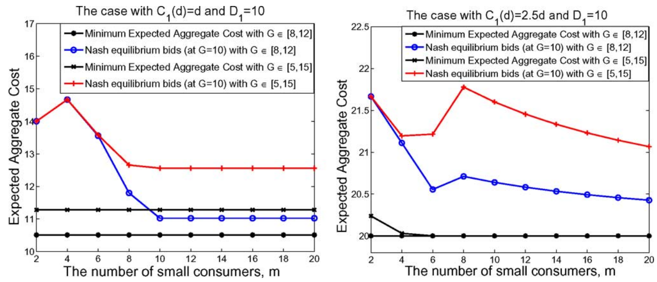

Fig. 3. Plot of expected aggregate cost resulting from the unique Nash equilibrium bids (at $G = 10$) and the minimum expected aggregate cost, with different m and distributions of the random G.

consumers play a Nash equilibrium strategy profile as if the total load adjustment were 10, the expected value of the random G.

The expected aggregate cost resulting from the Nash equilibrium bids (at $G = 10$) and the minimum expected aggregate cost are depicted in Fig. 3. Comparing Figs. 2 and 3, we conclude that the randomness in G does not significantly degrade the performance of the proposed market mechanism.[6] The efficiency loss resulting from the Nash equilibrium bids at $G = 10$ (depicted in Fig. 3) is comparable to that in the perfect information setting (depicted in Fig. 2), except that with uncertain G the efficiency loss does not converge to zero as m grows large; instead even for large m there is a (small) gap between the expected aggregate cost resulting from the Nash equilibrium bids (at $G = 10$) and the minimum expected aggregate cost (see Fig. 3).

VIII. Conclusion

We consider a model where a finite number of agents (i.e., consumers participating in a demand response program) compete to meet an inelastic demand for a single good (i.e., load adjustment). Each agent is characterized by a capacity limit and a cost function that is convex in the amount of load adjustment it provides (up to its capacity limit). We study the effect of agents' capacity limits on the efficiency of a Nash equilibrium, when every agent acts strategically to maximize her own payoff. In particular, we consider a simple uniform price market mechanism where the market clearing price is determined by the (parameterized) supply functions reported by all agents, and compare the aggregate cost resulting from a Nash equilibrium to the minimum possible cost.

We establish an upper bound on the ratio of the aggregate cost resulting from a Nash equilibrium to the minimum possible aggregate cost, which depends only on market participants' capacity constraints and the total (inelastic) demand. By restricting agents' strategy space to be one dimensional, the proposed market mechanism achieves approximate social optimality at a Nash equilibrium, if the capacity limit of all consumers (excluding the consumer with the largest capacity limit) is much larger than the total amount of load to be adjusted.

[6]This is intuitive because given the (Nash equilibrium) bids submitted by consumers, the allocation of consumers' load adjustment changes mildly with the variation of G.

Qualitatively similar results are established for a multi-stage setting with deferrable loads.

This paper serves as a starting point for designing practical and efficient demand response markets. Future work would involve comparisons between the proposed market mechanism and other simple bidding schemes (e.g., linear supply function bidding) under a variety of practical consumer cost functions.

Appendix A
Proof of Theorem 3.1

Given $p > 0$, each consumer's payoff function [see (5)]is concave. Therefore, an action vector is a competitive equilibrium, if and only if each of its components $b_i \in [0, p(D_i + L_i)]$, and it satisfies the following condition:

$$\begin{aligned} \frac{\partial^- C_i(S_i(b_i,p))}{\partial d_i} &\le p, \qquad \text{if} \quad 0 \le b_i < p(D_i+L_i) \\ \frac{\partial^+ C_i(S_i(b_i,p))}{\partial d_i} &\ge p, \qquad \text{if} \quad 0 < b_i \le p(D_i+L_i). \end{aligned} \tag{16}$$

Note that, at a competitive equilibrium $\mathbf{b}$, consumer i would never submit a bid that is larger than $p(D_i + L_i)$; in that case, consumer i obtains a negative payoff because $S_i(b_i, p) < -L_i$.

Since the objective function in (1) is convex, and the optimization problem is over a convex, compact feasible set, there exists an optimal solution. The following condition is necessary and sufficient for a feasible solution to (1), $\mathbf{d}^*$, to be optimal:

$$\begin{aligned} \frac{\partial^- C_i(d_i^*)}{\partial d_i} &\le \mu, \qquad \text{if} - L_i < d_i^* \le D_i; \\ \frac{\partial^+ C_i(d_i^*)}{\partial d_i} &\ge \mu, \qquad \text{if} - L_i \le d_i^* < D_i \end{aligned} \tag{17}$$

where μ is the Lagrange multiplier for the constraint $\sum_i d_i = G$. Since C_i is strictly increasing over $[-L_i, D_i]$ and at least one component of $\mathbf{d}^*$ is positive, it follows that $\mu > 0$. It is not hard to see that the action vector $\{b_i = \mu(D_i - d_i^*)\}$ with $p = \mu$ satisfies conditions (16), and is therefore a competitive equilibrium. On the other hand, for any competitive equilibrium, it is straightforward to show that the supply vector $\{D_i - b_i/p\}_{i=1}^n$, together with a μ value set equal to the price $p > 0$ satisfies the conditions (17), i.e., the resulting allocation is socially optimal.

Appendix B
Proof of Theorem 4.1

We first derive necessary and sufficient conditions for an action vector $\mathbf{b}$ to be a Nash equilibrium. We then show that there exists a unique optimal solution to the optimization problem in (7). In step 3), we derive necessary and sufficient optimality condition for the optimization problem in (7). The correspondence between the Nash equilibrium condition and the optimality condition establishes the existence of a Nash equilibrium and the uniqueness of the resulting allocation.

Step 1: Necessary and Sufficient Nash Equilibrium Condition: We argue in this step that a vector $\mathbf{b}$ is a Nash equilibrium, if and only if it has at least two positive components, each of its component $b_i \le c_i$ [c_i is a constant to be defined in (19)], and it satisfies the following condition:

$$\frac{\partial^- C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i}\left(1+\frac{S_i(b_i,p(\mathbf{b}))}{-G+\sum_{j\neq i} D_j}\right) \le p(\mathbf{b}), \quad \text{if } 0 \le b_i < c_i$$
$$\frac{\partial^+ C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i}\left(1+\frac{S_i(b_i,p(\mathbf{b}))}{-G+\sum_{j\neq i} D_j}\right) \ge p(\mathbf{b}), \quad \text{if } 0 < b_i \le c_i \tag{18}$$

where

$$c_i = \begin{cases} \dfrac{(D_i+L_i)\sum_{j\neq i} b_j}{-G-L_i+\sum_{j\neq i} D_j}, & \text{if } -G-L_i+\sum_{j\neq i} D_j > 0 \\ \infty, & \text{if } -G-L_i+\sum_{j\neq i} D_j \le 0. \end{cases} \tag{19}$$

We note that, if $-G-L_i+\sum_{j\neq i} D_j > 0$, then at a Nash equilibrium $\mathbf{b}$, consumer i would never submit a bid that is larger than

$$c_i = \frac{(D_i+L_i)\sum_{j\neq i} b_j}{-G-L_i+\sum_{j\neq i} D_j}.$$

This is because bidding c_i will always yield consumer i a higher payoff than submitting a bid larger than c_i. On the other hand, if $-G-L_i+\sum_{j\neq i} D_j \le 0$, we have $S_i(b_i, p(\mathbf{b})) \ge -L_i$ under all possible nonnegative vectors $\mathbf{b}$.

Let $\mathbf{b}$ be a Nash equilibrium. Lemma 4.2 shows that at least two components of $\mathbf{b}$ are positive, and that the market clearing price $p(\mathbf{b})$ is positive. We can therefore write consumer i's payoff as

$$\pi_i(b_i, \mathbf{b}_{-i}) = \frac{D_i(b_i + \sum_{j\neq i} b_j)}{-G+\sum_{j=1}^{n} D_j} - b_i - C_i\left(D_i - \frac{b_i(-G+\sum_{j=1}^{n} D_j)}{b_i+\sum_{j\neq i} b_j}\right) \tag{20}$$

which can be shown to be continuous and concave in b_i, over the domain $[0,\infty)$. Since for every i, $\pi_i(b_i, \mathbf{b}_{-i})$ is concave in b_i, the following condition is necessary and sufficient for a vector $\mathbf{b}$ to be a Nash equilibrium:

$$\frac{\partial^+ \pi_i(b_i, \mathbf{b}_{-i})}{\partial b_i} \le 0, \quad \text{if } 0 \le b_i < c;$$
$$\frac{\partial^- \pi_i(b_i, \mathbf{b}_{-i})}{\partial b_i} \ge 0, \quad \text{if } 0 < b_i \le c. \tag{21}$$

Substituting (20) into (21), we obtain

$$\frac{\partial^- C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i}\left(1-\frac{b_i}{\sum_j b_j}\right) \le \frac{-G+\sum_{j\neq i} D_j}{-G+\sum_j D_j} p(\mathbf{b}), \quad \text{if } 0 \le b_i < c$$
$$\frac{\partial^+ C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i}\left(1-\frac{b_i}{\sum_j b_j}\right) \ge \frac{-G+\sum_{j\neq i} D_j}{-G+\sum_j D_j} p(\mathbf{b}), \quad \text{if } 0 < b_i \le c. \tag{22}$$

Through the following relation:

$$1-\frac{b_i}{\sum_j b_j} = 1 - \frac{(D_i - S_i(b_i,p(\mathbf{b})))p(\mathbf{b})}{(\sum_j D_j - \sum_j S_j(b_j, p(\mathbf{b})))p(\mathbf{b})} = \frac{-G+\sum_{j\neq i} D_j + S_i(b_i,p(\mathbf{b}))}{\sum_j D_j - G}$$

it is straightforward to see that (22) is equivalent to (18).

Step 2: Existence and Uniqueness of an Optimal Solution to (7): In this step, we show that there exists a unique optimal solution to the optimization problem in (7). We first argue that $\hat{C}_i(d_i)$ is continuous, strictly convex and strictly increasing over $d_i \ge -L_i$. From the definition in (8) we have

$$\frac{\partial^- \hat{C}_i(d_i)}{\partial d_i} = \left(1+\frac{d_i}{-G+\sum_{j\neq i} D_j}\right)\frac{\partial^- C_i(d_i)}{\partial d_i};$$
$$\frac{\partial^+ \hat{C}_i(d_i)}{\partial d_i} = \left(1+\frac{d_i}{-G+\sum_{j\neq i} D_j}\right)\frac{\partial^+ C_i(d_i)}{\partial d_i}. \tag{23}$$

Since C_i is strictly increasing and convex, for any $-L_i \le d_i < d_i'$, we have

$$0 \le \frac{\partial^+ \hat{C}_i(d_i)}{\partial d_i} < \frac{\partial^- \hat{C}_i(d_i')}{\partial d_i'} \le \frac{\partial^+ \hat{C}_i(d_i')}{\partial d_i'}$$

which implies that $\hat{C}_i$ is strictly increasing and strictly convex over $[-L_i, \infty)$. Since, for every i, $\hat{C}_i$ is continuous and strictly convex, the optimization problem (7) over a convex, compact feasible region must have a unique optimal solution.

Step 3: Necessary and Sufficient Optimality Condition for the Problem in (7): Let $\mathbf{d} = (d_1, \dots, d_n)$ be the unique optimal solution to (7). There exists a Lagrange multiplier μ such that

$$\begin{aligned} &\left(1+\frac{d_i}{-G+\sum_{j\neq i} D_j}\right)\frac{\partial^- C_i(d_i)}{\partial d_i} \le \mu, \quad \text{if } -L_i < d_i \le D_i \\ &\left(1+\frac{d_i}{-G+\sum_{j\neq i} D_j}\right)\frac{\partial^+ C_i(d_i)}{\partial d_i} \ge \mu, \quad \text{if } -L_i \le d_i < D_i. \end{aligned} \tag{24}$$

Since at least one d_i is positive and C_i is strictly increasing, we have that $\mu > 0$. We now consider the action vector $\{b_i = (D_i - d_i)\mu\}_{i=1}^n$. Note that at least two components of $\mathbf{b}$ are positive, because $\sum_{j\neq i} D_j > G$ for every i. Since $d_i = D_i$ if and only if $b_i = 0$, and $d_i = -L_i$ if and only if $b_i = c_i$,[7] it is not hard to see from (24) that the action vector $\{(D_i - d_i)\mu\}_{i=1}^n$ satisfies the conditions in (18), and is therefore a Nash equilibrium.

Finally, we argue that all Nash equilibria result in the same demand response that is an optimal solution to (7). A Nash equilibrium $\mathbf{b}$ satisfies conditions (18). It follows that the vector $\{S_i(b_i, p(\mathbf{b}))\}_{i=1}^n$ satisfies condition (24), with $p(\mathbf{b}) > 0$ being the Lagrange multiplier. Since $\hat{C}_i$ is strictly convex for every i, conditions in (24) suffice that $\{S_i(b_i, p(\mathbf{b}))\}_{i=1}^n$ is an optimal solution to the problem in (7).

Step 4: Uniqueness of Nash Equilibrium Under an Additional Assumption: We are left to show the uniqueness of Nash equilibrium, under an additional assumption that all consumers have continuously differentiable cost functions. In this setting, the necessary and sufficient Nash equilibrium condition (18) can be written as

$$\begin{aligned} &C_i'(S_i(b_i, p(\mathbf{b})))\left(1+\frac{S_i(b_i, p(\mathbf{b}))}{-G+\sum_{j\neq i} D_j}\right) \le p(\mathbf{b}), \quad \text{if } 0 \le b_i < c_i \\ &C_i'(S_i(b_i, p(\mathbf{b})))\left(1+\frac{S_i(b_i, p(\mathbf{b}))}{-G+\sum_{j\neq i} D_j}\right) \ge p(\mathbf{b}), \quad \text{if } 0 < b_i \le c_i. \end{aligned} \tag{25}$$

Suppose that there are two distinct Nash equilibria $\mathbf{b}$ and $\mathbf{b}'$. We now prove the uniqueness of Nash equilibrium by considering the following two cases.

1) If there exists a consumer i such that $b_i \in (0, c_i)$ at the Nash equilibrium $\mathbf{b}$, then, according to (25), we have

$$C_i'(S_i(b_i, p(\mathbf{b})))\left(1+\frac{S_i(b_i, p(\mathbf{b}))}{-G+\sum_{j\neq i} D_j}\right) = p(\mathbf{b}).$$

We note also that consumer i's load adjustment $S_i(b_i, p(\mathbf{b}))$must lie in the interval $(-L_i, D_i)$. We have shown that consumer i must provide the same amount of load adjustment at the two Nash equilibria $\mathbf{b}$ and $\mathbf{b}'$, and as a result, $S_i(b_i', p(\mathbf{b}')) \in (-L_i, D_i)$. We therefore have

$$C_i'(S_i(b_i', p(\mathbf{b}')))\left(1+\frac{S_i(b_i', p(\mathbf{b}'))}{-G+\sum_{j\neq i} D_j}\right) = p(\mathbf{b}').$$

The above two equalities lead to $p(\mathbf{b}) = p(\mathbf{b}')$, which in turn implies that $\mathbf{b} = \mathbf{b}'$ [see (2)].

2) Suppose now there does not exist a consumer i such that $b_i \in (0, c_i)$. At the Nash equilibrium $\mathbf{b}$, each consumer i bids either c_i or 0, and provides either $-L_i$ or D_i load adjustment. In this case, the n consumers can be divided into two groups A and B, such that every consumer i in group A bids c_i and every consumer j in group B bids 0. From the condition in (25), we have

$$C_j'(D_j)\left(1+\frac{D_j}{-G+\sum_{k\neq j} D_k}\right) \le p(\mathbf{b}) \le C_i'(-L_i)\left(1+\frac{-L_i}{-G+\sum_{k\neq i} D_k}\right)$$

for every $i \in A$ and every $j \in B$. It is straightforward to check [from condition (25)]that at any Nash equilibrium the market price must lie in the following range:

$$\left(\max_{j\in B} C_j'(D_j)\frac{-G+\sum_k D_k}{-G+\sum_{k\neq j} D_k}, \min_{i\in A} C_i'(-L_i)\frac{-G-L_i+\sum_{k\neq i} D_k}{-G+\sum_{k\neq i} D_k}\right)$$

otherwise the net supply would not be G. It follows that $\mathbf{b}$ is the unique Nash equilibrium.

APPENDIX C
PROOF OF PROPOSITION 4.1

We note that consumer i achieves a zero payoff if her load adjustment is zero. This implies that a consumer must obtain at least a zero payoff at a Nash equilibrium, because given any bids

[7] If $-G - L_i + \sum_{j\neq i} D_j \le 0$, $d_i = -L_i$ is not feasible, and b_i is always less than $c_i = \infty$.

submitted by other consumers, a consumer can always choose a bid that yields her zero supply.

Let $\mathbf{b}$ be a Nash equilibrium. Suppose that consumer i's supply is positive at the equilibrium, i.e., $S_i(b_i, p(\mathbf{b})) > 0$. It follows from the equilibrium condition in (18) (see the proof of Theorem 4.1 in Appendix B) that

$$\frac{\partial^- C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i} < p(\mathbf{b}).$$

Since C_i is a convex function, we have

$$0 < C_i(S_i(b_i, p(\mathbf{b}))) \le \frac{\partial^- C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i} S_i(b_i, p(\mathbf{b})) < p(\mathbf{b}) S_i(b_i, p(\mathbf{b}))$$

which implies that consumer i achieves a positive payoff at the equilibrium. On the other hand, if $S_i(b_i, p(\mathbf{b})) < 0$, it follows from (18) that

$$\frac{\partial^+ C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i} > p(\mathbf{b}).$$

We similarly have

$$C_i(S_i(b_i, p(\mathbf{b}))) \le \frac{\partial^+ C_i(S_i(b_i, p(\mathbf{b})))}{\partial d_i} S_i(b_i, p(\mathbf{b})) < p(\mathbf{b}) S_i(b_i, p(\mathbf{b})) < 0.$$

Appendix D
Proof of Proposition 4.2

Since $\{d_j\}_{j=1}^n$ is the supply vector corresponding to a Nash equilibrium with the capacity limit $\{D_j\}_{j=1}^n$ and $d_i < D_i$, it follows from (24) that

$$\left(1 + \frac{d_i}{-G + \sum_{j \ne i} D_j}\right) \frac{\partial C_i(d_i)}{\partial d_i} \ge \mu.$$

We will show the proposition by contradiction. Suppose that $\bar{d}_i > d_i$. Since $\{\bar{d}_j\}_{j=1}^n$ is the supply vector corresponding to a Nash equilibrium with the capacity limit $\{\bar{D}_j\}_{j=1}^n$ and $-L_i \le d_i < \bar{d}_i$, it follows from (24) that

$$\left(1 + \frac{\bar{d}_i}{-G + \sum_{j \ne i} \bar{D}_j}\right) \frac{\partial C_i(\bar{d}_i)}{\partial \bar{d}_i} \le \bar{\mu}.$$

Since $D_j = \bar{D}_j$ for every $j \ne i$ and $\bar{d}_i > d_i$, we have $\bar{\mu} > \mu$. Since $\sum_{j=1}^n d_j = \sum_{j=1}^n \bar{d}_j = G$, there must exist some consumer k such that $\bar{d}_k < d_k$. Since $-L_k \le \bar{d}_k < d_k \le D_k$, it follows from (24) that

$$\left(1 + \frac{d_k}{-G + \sum_{j \ne k} D_j}\right) \frac{\partial C_k(d_k)}{\partial d_k} \le \mu$$

$$\left(1 + \frac{\bar{d}_k}{-G + \sum_{j \ne k} \bar{D}_j}\right) \frac{\partial C_k(\bar{d}_k)}{\partial \bar{d}_k} \ge \bar{\mu}.$$

Since $\sum_{j \ne k} D_j < \sum_{j \ne k} \bar{D}_j$ and $d_k > \bar{d}_k$, the above two inequalities imply that $\bar{\mu} < \mu$. We therefore conclude that $\bar{d}_i \le d_i$.

Appendix E
Proof of Theorem 5.1

We first show that the virtual cost function $\hat{C}_j$ defined in (8) is larger than the original cost function C_j. Since C_j is nondecreasing, for $j = 1, \ldots, n$ and $d \in [0, \min\{D, D_j\}]$, we have

$$\hat{C}_j(d) \ge \left(1 + \frac{d}{-G + \sum_{k \ne j} D_k}\right) C_j(d) - \frac{1}{-G + \sum_{k \ne j} D_k} \int_0^d C_j(x) dx = C_j(d).$$

Similarly, for $j = 1, \ldots, n$ and $d \in [-L_j, 0)$, we have

$$\hat{C}_j(d) \ge \left(1 + \frac{d}{-G + \sum_{k \ne j} D_k}\right) C_j(d) + \frac{1}{-G + \sum_{k \ne j} D_k} \int_d^0 C_j(x) dx = C_j(d).$$

On the other hand, since $C_j(d) \ge 0$ for$d \ge 0$, we have

$$\hat{C}_j(d) \le \left(1 + \frac{d}{-G + \sum_{k \ne j} D_k}\right) C_j(d), \qquad d \ge 0.$$

It follows that, for $j = 1, \ldots, n$ and $d \in [0, \min\{G, D_j\}]$, we have

$$\hat{C}_j(d) \le \left(1 + \frac{\min\{D_j, D\}}{-G + \sum_{k \ne j} D_k}\right) C_j(d). \tag{26}$$

Let $\mathbf{d}^*$ be the nonnegative socially optimal allocation, and $\mathbf{d}$ be an allocation resulting from a Nash equilibrium, respectively. We have

$$\begin{aligned}\sum_{j=1}^{n} C_j(d_j) &\leq \sum_{j=1}^{n} \hat{C}_j(d_j)\\ &\leq \sum_{j=1}^{n} \hat{C}_j(d_j^*)\\ &\leq \sum_{j=1}^{n}\left(1+\frac{\min\{D_j,G\}}{-G+\sum_{k\neq j} D_k}\right) C_j(d_j^*)\\ &\leq \left(1+\frac{\min\{D_i,G\}}{-G+\sum_{k\neq i} D_k}\right)\sum_{j=1}^{n} C_j(d_j^*)\end{aligned}$$

where the first inequality is true because $\hat{C}_j(d_j) \leq C_j(d_j)$ for every j and every d_j, the second inequality follows from the fact that $\mathbf{d}$ minimizes the sum of virtual cost functions $\hat{C}_j$ [see the optimization problem in (7)], the third inequality follows from (26), and last inequality is true because $D_i = \max_j\{D_j\}$.

It remains to show that the bound is tight for the case $G \leq D_i$. Fixing $G > 0$ and $n \geq 2$, we consider a model where $D_1 \geq D_2 = \cdots = D_n$, and $L_j = 0$ for every j. Let r be a positive constant such that $G/n < r < G \leq D_1$, and let $\delta \in (0,1)$. Consumer 1's cost function is

$$C_1(d_1) = \begin{cases} \delta d_1, & \text{if } 0 \leq d_1 \leq r,\\ d_1 - r + \delta r, & \text{if } r < d_1 \leq D_1\end{cases}$$

and for $j = 2,\ldots,n$,

$$C_j(d_j) = \alpha d_j, \qquad d_j \geq 0$$

where

$$\alpha = \frac{1+\dfrac{r}{-G+\sum_{k=2}^{n} D_k}}{1+\dfrac{G-r}{(-G+\sum_{k\neq 2}^{n} D_k)(n-1)}}. \tag{27}$$

Since $G/n < r$ and $D_1 = \max_k\{D_k\}$, it follows that $\alpha > 1$. Therefore, a socially optimal allocation is given by

$$d_1^* = G; \qquad d_j^* = 0, \quad j \geq 2.$$

We now argue that the supply vector

$$\mathbf{d} = (d_1,\ldots,d_n) = \left(r, \frac{G-r}{n-1},\ldots,\frac{G-r}{n-1}\right)$$

is an optimal solution to (7). To see this, let $\mu = 1 + r/(-G + \sum_{k=2}^{n} D_k)$, and we have

$$\left(1+\frac{d_1}{-G+\sum_{k\neq 1} D_k}\right)\frac{\partial^- C_1(d_1)}{\partial d_1} = \delta\mu \leq \mu;$$

$$\left(1+\frac{d_1}{-G+\sum_{k\neq 1} D_k}\right)\frac{\partial^+ C_1(d_1)}{\partial d_1} = \mu;$$

$$\left(1+\frac{d_j}{-G+\sum_{k\neq j} D_k}\right)\frac{\partial C_j(d_j)}{\partial d_j} = \mu, \qquad j = 2,\ldots,n$$

where the last inequality follows from (27). Since the preceding condition is equivalent to condition (24), it follows that the vector $\mathbf{d}$ is an optimal solution to (7), and is therefore the allocation resulting from a Nash equilibrium. At the Nash equilibrium, the aggregate utility loss is

$$\sum_j C_j(d_j) = \delta r + \alpha(G-r)$$

while at a social optimum we have

$$\sum_j C_j(d_j^*) = G - r + \delta r.$$

We obtain

$$\frac{\sum_j C_j(d_j)}{\sum_j C_j(d_j^*)} = \frac{\delta r + \alpha(G-r)}{G-r+\delta r}.$$

Let $r \to G$, and $\delta r/(G-r) \to 0$, e.g., $\delta = (G-r)^2$. The preceding ratio converges to α, whose limit is given by

$$\lim_{r\to G}\frac{1+\dfrac{r}{-G+\sum_{k=2}^{n} D_k}}{1+\dfrac{G-r}{(-G+\sum_{k\neq 2}^{n} D_k)(n-1)}} = 1+\frac{G}{-G+\sum_{k=2}^{n} D_k}.$$

Appendix F
Proof of Theorem 6.1

We will prove the theorem through an approach similar to that used in the proof of Theorem 4.1.

Step 1: Necessary and Sufficient Nash Equilibrium Condition: From (11) we have

$$\frac{b_i}{\sum_j b_j} = \frac{D_{i,t} - d_{i,t}}{\sum_j (D_{j,t} - d_{j,t})} = \frac{D_{i,t} - d_{i,t}}{\sum_j D_{j,t} - G_t}, \qquad \forall t,$$

which implies that

$$\frac{\sum_{j\neq i} b_j}{\sum_j b_j} = 1 - \frac{b_i}{\sum_j b_j} = 1 - \frac{\sum_t (D_{i,t} - d_{i,t})}{\sum_t (\sum_j D_{j,t} - G_t)} = 1 - \frac{D_i - d_i}{\sum_t R_t} \tag{28}$$

where $D_i = \sum_t D_{i,t}, d_i = \sum_t d_{i,t}$, and $R_t = -G_t + \sum_j D_{j,t}$. We let $p_t(\mathbf{b})$ denote the price at stage t, under an action vector $\mathbf{b}$. Following (12), we have

$$\sum_t \left(\frac{1}{p_t(\mathbf{b})}\right) = \sum_t \frac{-G_t + \sum_{i=1}^{n} D_{i,t}}{\sum_{j=1}^{n} b_j}. \tag{29}$$

Based on the relations in (28) and (29), we obtain necessary and sufficient equilibrium conditions by differentiating the payoff function in (13) with respect to b_i. A vector $\mathbf{b}$ is a Nash equilibrium, if and only if it has at least two positive components, each of its component $b_i \leq c_i$ [c_i is a constant to be defined in (31)], and it satisfies the following condition:

$$\begin{aligned} &\frac{\partial^- C_i(d_i(\mathbf{b}))}{\partial d_i(\mathbf{b})} \frac{1 - \frac{(D_i - d_i)}{(\sum_t R_t)}}{T - \sum_t \left(\frac{D_{i,t}}{R_t}\right)} \leq \frac{1}{\sum_t \frac{1}{p_t(\mathbf{b})}}, \quad \text{if } 0 \leq b_i < c_i \\ &\frac{\partial^- C_i(d_i(\mathbf{b}))}{\partial d_i(\mathbf{b})} \frac{1 - \frac{(D_i - d_i)}{(\sum_t R_t)}}{T - \sum_t \left(\frac{D_{i,t}}{R_t}\right)} \geq \frac{1}{\sum_t \frac{1}{p_t(\mathbf{b})}}, \quad \text{if } 0 < b_i \leq c_i \end{aligned} \tag{30}$$

where $d_i(\mathbf{b})$ is the aggregate load adjustment provided by consumer i under the action vector $\mathbf{b}$, and

$$c_i = \begin{cases} \frac{(D_i + L_i)\sum_{j\neq i} b_j}{-D_i - L_i + \sum_t R_t}, & \text{if } D_i + L_i < \sum_t R_t, \\ \infty, & \text{if } D_i + L_i \geq \sum_t R_t. \end{cases} \tag{31}$$

We note that, if $D_i + L_i < \sum_t R_t$, then at a Nash equilibrium $\mathbf{b}$, consumer i would never submit a bid that is larger than

$$c_i = \frac{(D_i + L_i)\sum_{j\neq i} b_j}{-D_i - L_i + \sum_t R_t}$$

because a bid that is larger than c_i yields the consumer an aggregate load adjustment $d_i(\mathbf{b}) < -L_i$, and she could have obtained a higher payoff by bidding c_i and providing $-L_i$ amount of total load adjustment. On the other hand, if $D_i + L_i \geq \sum_t R_t$, we have $d_i(\mathbf{b}) \geq -L_i$ under all possible nonnegative vectors $\mathbf{b}$.

Step 2: Necessary and Sufficient Optimality Condition for the Problem in (14): Since for every i, $\hat{C}_i(d_i)$ is continuous, strictly convex and strictly increasing over $d_i \geq -L_i$, the optimization problem (14) over a convex, compact feasible region must have a unique optimal solution. Let $\mathbf{d} = (d_1, \ldots, d_n)$ be the unique optimal solution to (14). Since each cost function $C_i(\cdot)$ is strictly increasing, there exists a positive Lagrange multiplier μ such that

$$\begin{aligned} &\frac{\partial^- C_i(d_i)}{\partial d_i} \frac{1 - \frac{(D_i - d_i)}{(\sum_t R_t)}}{T - \sum_t \left(\frac{D_{i,t}}{R_t}\right)} \leq \mu, \\ &\qquad \text{if} - L_i < d_i \leq D_i \\ &\frac{\partial^- C_i(d_i)}{\partial d_i} \frac{1 - \frac{(D_i - d_i)}{(\sum_t R_t)}}{T - \sum_t \left(\frac{D_{i,t}}{R_t}\right)} \geq \mu, \\ &\qquad \text{if} - L_i \leq d_i < D_i. \end{aligned} \tag{32}$$

Given the optimal solution to (14), $(d_1, \ldots, d_n)$, the action vector $\{(D_i - d_i)/(T\mu)\}_{i=1}^n$ satisfies the conditions in (30)and is therefore a Nash equilibrium.

We now argue that all Nash equilibria yield a unique aggregate load adjustment vector that is an optimal solution to (14). A Nash equilibrium $\mathbf{b}$ satisfies conditions (30). It follows that the vector $\{d_i(\mathbf{b})\}_{i=1}^n$ satisfies condition (32), with $\sum_t 1/p_t(\mathbf{b}) > 0$ being the Lagrange multiplier. Since $\hat{C}_i$ is strictly convex for every i, conditions in (32) suffice that $\{d_i(\mathbf{b})\}_{i=1}^n$ is an optimal solution to the problem in (14).

Appendix G
Proof of Theorem 6.2

It follows from (15) that, for every j and every $d_j \in [-L_j, D_j]$, we have

$$\hat{C}_j(d_j) \geq \frac{1 - \frac{D_j}{(\sum_t R_t)}}{T - \sum_t \left(\frac{D_{j,t}}{R_t}\right)} C_j(d_j)$$

$$= \frac{1}{T} \cdot \frac{1 - \dfrac{D_j}{(\sum_t R_t)}}{1 - \sum_t \dfrac{D_{j,t}}{(TR_t)}} C_j(d_j)$$
$$\geq \frac{1}{T} C_j(d_j).$$

Here, the last inequality follows from the fact that

$$\frac{\sum_t D_{j,t}}{\sum_t R_t} \leq \sum_{t=1}^{T} \frac{D_{j,t}}{TR_t} < 1$$

where the second inequality must strictly hold, because of Assumption 6.1, i.e., $G_t > D_{j,t}$ for every j. It is also straightforward to check from (15) that

$$\hat{C}_j(d_j) \leq \frac{1 - \dfrac{(D_j - d_j)}{(\sum_t R_t)}}{T - \sum_t (\dfrac{D_{j,t}}{R_t})} C_j(d_j) \quad \forall\, d_j \in [0, D_j]. \tag{33}$$

For $j = 1, \ldots, n$, let $d_j^* \geq 0$ denote the total load adjustment of consumer j at an optimal solution to (9). We have

$$\begin{aligned}
\sum_{j=1}^{n} C_j(d_j) &\leq T \sum_{j=1}^{n} \hat{C}_j(d_j) \\
&\leq T \sum_{j=1}^{n} \hat{C}_j(d_j^*) \\
&\leq \sum_{j=1}^{n} \frac{1 - \dfrac{(D_j - d_j)}{(\sum_t R_t)}}{1 - \sum_t \dfrac{D_{j,t}}{(TR_t)}} C_j(d_j^*) \\
&\leq \sum_{j=1}^{n} \frac{1}{1 - \sum_t \dfrac{D_{i^t,t}}{(TR_t)}} C_j(d_j^*) \\
&= \frac{1}{1 - \sum_t \dfrac{D_{i^t,t}}{(TR_t)}} \sum_{j=1}^{n} C_j(d_j^*)
\end{aligned}$$

where the second inequality follows from the fact that $\mathbf{d}$ minimizes the sum of virtual cost functions $\hat{C}_j$ [see the optimization problem in (14)], the third inequality follows from (33), and fourth inequality is true because $D_j - d_j \geq 0$ and $D_{j,t} \leq D_{i^t,t}$ for every j and every t.

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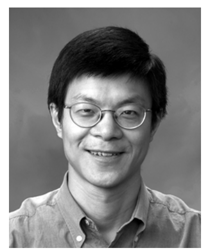

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