Estimating the Probability of Infeasible Real-Time Dispatch Without Exact Distributions of Stochastic Wind Generations

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***Abstract*—This paper proposes a data-driven and convex optimization based method to quantify the probability of infeasible real-time dispatch (RTD) of power systems with volatile wind energy integrations. The required information about wind power is a finite sequence of moments, instead of the exact probability distribution function (PDF). The candidate PDFs are restricted in a functional set subject to moment constraints. By assuming the dispatchable region of nodal wind power injection is available, we propose a semi-definite programming (SDP) based method and a linear programming (LP) based method to estimate the probability of infeasibility in the worst wind power distribution. We also suggest two alternative methods based on the emerging generalized Chebyshev inequality (GCI) and generalized Gauss inequality (GGI), which only utilize the first and second order moments, and boil down to solving SDPs. We compare the performances of all the discussed methods on the moderately sized IEEE 118-bus system. Experimental results demonstrate that our method can offer monotonically better estimation when higher order moments are provided and is competitive with GCI and GGI.**

***Index Terms*—Convex optimization, power system operation, uncertainty quantification, wind generation.**

NOMENCLATURE

The major symbols and notations used throughout the paper are defined below for quick reference. Others are defined after their first appearances as required.

Parameters:

a_i, b_i	Energy production cost coefficients of generator i.
C_j^W	Capacity of wind farm j, $C^W = \{C_j^W\}, \forall j$.
C^R	Available cost of real-time dispatch.
d_i^+	Up-regulation cost coefficient of generator i.
d_i^-	Down-regulation cost coefficient of generator i.
F_l	Transmission capacity of line l.
H, h	Matrix coefficient of the dispatchable region.
P_i^l	Minimal output of generator i.
P_i^u	Maximal output of generator i.
p_i^f	Current output of generator i, $p^f = \{p_i^f\}, \forall i$
p_q	Power demand of load q.
N_W	Number of wind farms.
R_i^+	Ramp-up limit of generator i.
R_i^-	Ramp-down limit of generator i.
Δt	Time duration of the current dispatch interval.
w_j^e	Current output of wind farm j, $w^e = \{w_j^e\}, \forall j$.
w_j^B	Base value of wind farm j, $w^B = \{w_j^B\}, \forall j$.
π_{il}	Power Transfer Distribution Factor (PTDF) from generator i to line l.
π_{jl}	PTDF from wind farm j to line l.
π_{ql}	PTDF from load q to line l.
σ_j	Square root variance of the output forecast error in wind farm j.
Θ	Covariance matrix of wind generation forecast error.

Decision Variables:

p_i^+	Up-regulation power of generator i .
p_i^-	Down-regulation power of generator i.
$p^\pm$	Real-time dispatch strategy, $p^\pm = \{p_i^+, p_i^-\}, \forall i$.
w_j	Actual output of wind farm j, $w = \{w_j\}, \forall j$, variable of the PDF as well as the dispatchable region.
Δw_j	Forecast error of wind farm j, $\Delta w = \{\Delta w_j\}, \forall j$, variable of the PDF and the dispatchable region.

Sets:

E_B^W	Hypercube of wind generation capability.
E_R^W	Finite samples of E_B^W.
$\mathbb{R}^n$	Set of n-dimensional real vector.
$\mathbb{S}^n$	Set of $n \times n$ symmetric matrix.
W^D	Dispatchable region of wind generation.

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$\bar{W}^D$	Non-dispatchable region of wind generation.
$\bar{W}_R^D$	Finite samples of $\bar{W}^D$.
$\mathbb{Z}^+$	Set of nonnegative integers.
$\Sigma(w)$	Set of positive polynomials in variable w that admits a sum-of-squares (SOS) decomposition.

Abbreviations:

ARO	Adjustable Robust Optimization.
GCI	Generalized Chebyshev Inequality.
GGI	Generalized Gauss Inequality.
GPI	Generalized Probability Inequality.
LP	Linear Programming.
MCS	Monte Carlo Simulation.
MILP	Mixed Integer Linear Programming.
PDF	Probability Distribution Function.
PTDF	Power Transfer Distribution Factor.
PSD	Positive Semi-definite.
RTD	Real-Time Dispatch.
SDP	Semi-definite Programming.
SOS	Sum-of-Squares.
TED	Traditional Economic Dispatch.

I. INTRODUCTION

THE large-scale utilization of renewable energies, such as wind, can remarkably reduce the dependence on fossil fuel and thereby help alleviate air pollution. But their variability also increases risks of power system dispatch infeasibility, in which case the generation would fail to balance the load, or the transmission network would be congested [1].

In many applications, the variable nature of the wind generation is described by a probability distribution function (PDF). Typical PDFs of the wind speed are summarized in [2] and [3]. The PDFs of wind power can be analytically derived using the relationship between wind speed and wind power [4]. Nevertheless, in short-term dispatch problems, the Gaussian distribution is also eligible to approximate the forecast error of wind power [5]–[7]. Given the PDF of wind power, optimal generation scheduling problems can be formulated using the stochastic programming approaches [5]–[11]. The reliability of system operation in the perspective of generation adequacy can also be evaluated by Monte Carlo simulation, please refer to [12]–[14] and references therein for details.

As indicated by [2] and [3], the exact PDFs of wind speed and wind power may be difficult to acquire. One can only seek approximations that best fit to the historical data. Recently, adjustable robust optimization (ARO) appears to be a good alternative for power system dispatch problems without exact PDFs of wind generation, applications have been found in the unit commitment [15]–[18], energy and reserve scheduling [19]–[21], optimal power flow [22], and reactive power optimization [23]. In ARO, uncertainties are modeled as a pre-specified uncertainty set. The dispatch strategy guarantees the existence of a feasible RTD as long as the uncertain data does not step outside the boundaries of the uncertainty set.

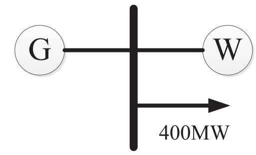

Fig. 1. A single-bus system.

However, the reliability of operation cannot be easily evaluated in the absence of PDFs, because one can hardly procure scenarios or time series that reflect the random behavior of renewable generations.

By assuming some actual data (may not be enough to claim the PDF) of wind generation is available, we assume the distribution of wind power is ambiguous, all candidate PDFs are restricted in a functional set, and should have moments consistent with those recovered from actual data. We will discuss how to estimate the likelihood of an infeasible RTD in the worst-case wind distribution. The simplest case of our problem is illustrated in Fig. 1. The system consists of a conventional generator, a wind farm and a load with 400 MW deterministic demand. The current output of generator and wind farm is equal to 200 MW, respectively. The wind generation is stochastic and its PDF unknown. We only have its forecast (200 MW, first-order moment) and variance (σ^2, second-order center moment) in the next one hour. When the wind generation deviates from 200 MW, the generator will respond to the actual wind power by adjusting its output so as to balance the demand. Because the ramping speed of the generator is limited, say 100 MW/h, the balancing condition cannot be maintained if $|w - 200| > 100$. According to Chebyshev inequality [24], the probability of infeasibility is

$$\Pr(|w - 200| > 100) \leq \frac{\sigma^2}{100^2}.$$

This inequality holds true regardless of the distribution of the random variable w. Recent advances in [25]–[28] raise optimization based approaches to compute the probability of certain event that is described by a multi-dimensional generalized probability inequality (GPI) subject to moment constraints and structured property, such as unimodality.

In this paper, we study the probability of infeasible RTD in more general settings that involve multiple wind farms, the power transmission network, and higher order moments of wind power variations. Combining the theory of dispatchable region in [29]–[31] and GPI proposed in [26], we propose the primal and dual formulations of the probability estimation problem. Since the non-dispatchable region that causes infeasibility is not a semi-algebraic set, the algorithm in [26] does not apply to our problem. By exploiting the polyhedral feature of the dispatchable region, we propose an SDP approximation and an LP relaxation of the dual problem to estimate the upper bound and lower bound of infeasible probability in the worst wind power distribution. We also discuss how to use GCI [25] and GGI [28] to compute this probability.

Note that our problem is an assessment rather than a strategic economic dispatch. We assume the current operating status is known, RTD constraints are projected onto the uncertainty subspace, yield the dispatchable region [29], and generator variables are eliminated in this process. So our method will not offer any generation plan, instead, it informs the operator how reliable the RTD is, and can help the operator make better dispatch decision indirectly. Our problem is also different from the generation adequacy oriented reliability evaluation, which usually involves longer timescales and contingencies such as line tripping and generator outages. Generation adequacy is only one requirement of RTD, there are substantially many other operating constraints to be met in RTD, such as the ramping limits of generators and security constraints of the transmission network. Meanwhile, we do not consider contingencies in RTD because the time scale is short. Such contingencies can be considered in the unit commitment [18] and economic dispatch [11], [21].

To the best of our knowledge, this is the first work that deals with this particular probability problem without exact PDFs of renewable generation. Furthermore, even provided with the exact PDFs of renewable generation, one still cannot compute such a probability in an analytical way, because it comes down to a multi-dimensional integration over a complicated polytope, the only way seems to be the Monte Carlo simulation. In view of the moderate data requirement and computational advantage, we believe our method would potentially become an important and promising method to quantify the impact of uncertainty on power system operation.

The remaining parts of this paper are organized as follows. The description of RTD, the primal and dual formulations of the probability problem are presented in Section II. The SDP and LP based bounding procedure is described in Section III. Case studies on the IEEE 118-bus system are reported in Section IV. Conclusions are given in Section V. The GCI and GGI based approaches are briefly introduced in the Appendix.

II. Mathematical Formulation

A. Model of the Real-Time Dispatch

Notations used in this section are defined in the Nomenclature. Suppose the current wind generation w^e is known and the conventional generation p^f has been scheduled with respect to w^e. The RTD seeks a valid corrective action in response to the actual wind generation w, so as to maintain the load balance and other operating constraints when $w \neq w^e$. This comes down to finding a feasible re-dispatch $p^{\pm} = \{p_i^+, p_i^-\}, \forall i$ in the following linear constraint set

$$P_i^l \leq p_i^f + p_i^+ - p_i^- \leq P_i^u, \forall i \quad (1.1)$$

$$\sum_i (p_i^f + p_i^+ - p_i^-) + \sum_j w_j = \sum_q p_q \quad (1.2)$$

$$-F_l \leq \sum_i \pi_{il}(p_i^f + p_i^+ - p_i^-) + \sum_j \pi_{jl} w_j - \sum_q \pi_{ql} p_q \leq F_l, \ \forall l \quad (1.3)$$

$$0 \leq p_i^+ \leq R_i^+ \Delta t, \ 0 \leq p_i^- \leq R_i^- \Delta t, \ \forall i \quad (1.4)$$

$$\sum_i (d_i^+ p_i^+ + d_i^- p_i^-) \leq C^R \quad (1.5)$$

where $p_i^f + p_i^+ - p_i^-$ is the output of conventional unit i after corrective action is deployed; constraint (1.1) is the generation capacity of conventional unit; constraint (1.2) is the system power balancing condition; constraint (1.3) restricts the power flow in each transmission line within its security limit; constraint (1.4) stipulates the regulation power of unit i within its ramping limit. By assuming the upward regulation and downward regulation are charged at different prices and imposing an available cost C^R of these corrective actions [30], constraint (1.5) further restricts the output range of generators in RTD. While one can certainly minimize the cost, here we consider the cost as a constraint because we will investigate the impact of C^R on the reliability of system operation later. The feasible region defined by constraints (1.1)–(1.5) can be arranged as a compact polytope shown below

$$\Lambda(p^f) = \{p^{\pm}, w | Bp^{\pm} + Cw \leq b - Ap^f\} \quad (2.1)$$

where matrices A, B, C and vector b correspond to the coefficients in constraints (1.1)–(1.5). Thus the feasible region of corrective action $p^{\pm}$ under given p^f and realized w can be defined in a compact form as

$$Y(p^f, w) = \{y^{\pm} \mid By^{\pm} \leq b - Ap^f - Cw\}. \quad (2.2)$$

Definition 1: RTD is feasible if and only if $Y(p^f, w) \neq \emptyset$.

In power system operation, we are interested in the region that consists of all w such that RTD is feasible, which inspires the following definition.

Definition 2: The dispatchable region W^D is the union of all admissible nodal wind power injection w, such that RTD is feasible, or mathematically, $W^D = \{w \mid Y(p^f, w) \neq \emptyset\}$.

The dispatchable region of wind generation is proposed in [29]. It is the projection of polytope $\Lambda(p^f)$ onto w-subspace and is also a polytope [29], which has a form of

$$W^D = \{w \in \mathbb{R}^{N_W} | Hw \leq h\}. \quad (3)$$

For bulk power systems with centralized wind power integration, the matrix coefficients H and h can be effectively computed by using the method in [30]. If the re-dispatch is restricted to be linearly dependent on the wind power forecast error $\Delta w = w - w^e$, W^D has a closed-form expression [31]. In this paper, we also use the non-dispatchable region defined as

$$\bar{W}^D = \{w \mid w \notin W^D\} \quad (4)$$

which is complementary to W^D on $\mathbb{R}^{N_W}$, and consists of all nodal wind power injections that will cause infeasibility. Strictly speaking, the non-dispatchable region $\bar{W}^D$ does not contain the boundaries of W^D and is an open set. Here we clarify in the following context we will include the boundary of W^D in $\bar{W}^D$ to facilitate developing tractable reformulations. Nevertheless, this will not influence the result of the probability problem, because the boundary set ∂W^D has zero measure on $\mathbb{R}^{N_W}$.

Sets W^D and $\bar{W}^D$ depend on the current dispatch strategy p. In this paper, we are not aiming at computing an optimal generation plan p^f. We assume p^f is provided by some economic dispatch method, such as the traditional economic dispatch (TED), the stochastic optimization based economic dispatch [11], or the ARO based economic dispatch [19], [20], [22]. For the purpose of simplification, we use the following TED to retrieve the current p^f, although it may not be the best choice in practice

$$\min \sum_i \left[a_i^2 (p_i^f)^2 + b_i p_i^f\right] \tag{5.1}$$

$$s.t.\ P_i^l \leq p_i^f \leq P_i^u,\ \forall i \tag{5.2}$$

$$\sum_i p_i^f + \sum_j w_j^e = \sum_q p_q \tag{5.3}$$

$$-F_l \leq \sum_i \pi_{il} p_i^f + \sum_j \pi_{jl} w_j^e - \sum_q \pi_{ql} p_q \leq F_l, \forall l \tag{5.4}$$

where objective (5.1) minimizes the production cost; constraint (5.2) is the generation capacity limitation; constraints (5.3) and (5.4) are the power balancing condition and line flow restriction with respect to the current wind generation w^e. TED (5.1)–(5.4) is a convex quadratic program with linear constraints, and can be solved by most off-the-shelf solvers.

B. Primal and Dual Formulation of the Probability Problem

For the purpose of notation brevity and clarity, let $\kappa = [k_1, k_2, \ldots, k_{N_W}]$ be a vector index with N_W entries, where N_W is the number of wind farms, $k_j \in \mathbb{Z}^+$, $j = 1, 2, \ldots, N_W$; Index set $J_k = \{\kappa | 1^T \kappa = k\}$ will be used to define k-th order moment; the notation $w^\kappa = w_1^{k_1} w_2^{k_2} \cdots w_{N_W}^{k_{N_W}}$, where $w = [w_1, w_2, \ldots w_{N_W}]^T$, will be frequently used in the rest parts of this paper; The k-th order moment of wind generation is the sequence $\sigma_\kappa = E[w^\kappa], \forall \kappa \in J_k$, where $E[\,]$ is the expectation operator; If wind generation $w_j, \forall j$ is independent, the moment $\sigma_\kappa = E[w_1^{k_1}] E[w_2^{k_2}] \cdots E[w_J^{k_J}], \forall \kappa \in J_k$ can be easily retrieved from the local data of each wind farm, otherwise, the sampled data should be synchronized in the presence of correlation; Index set $J(K) = \bigcup_{k=0}^{K} J_k$.

Suppose the PDF $f(w)$ of wind generation is not known exactly. Nevertheless, we can compute the moments of wind power up to some order from statistic analysis, and the PDF should provide moments consistent with those recovered from actual data. In view of this, $f(w)$ belongs to the functional set Ω_M^K subject to moment constraints up to K-th order

$$\Omega_M^K = \left\{ f(w) \;\middle|\; \begin{array}{c} \int_{w \in E_B^W} w^\kappa f(w) \mathrm{d}w = \sigma_\kappa, \forall \kappa \in J(K) \\ f(w) \geq 0,\ \forall w \in \mathbb{R}^{N_W} \end{array} \right\} \tag{6}$$

where $E_B^W = \{w \in \mathbb{R}^{N_W} \mid 0 \leq w \leq C^W\}$, in practice, the output of a wind farm can neither become negative, nor exceed its capacity C_j^W, so we restrict the integration in each moment constraints within the hypercube E_B^W. We impose non-negativity on $f(w)$, and define $\sigma_{0,\ldots,0} = 1$ in order to make sure that $\int_{w \in E_B^W} f(w) \mathrm{d}w = 1$. The remaining constraints in the first equation of Ω_M^K define each order moment of w.

In this paper, we consider how likely the RTD would be infeasible in the worst wind power distribution, yielding the following GPI

$$Z_P = \max \Pr[w \in \bar{W}^D] = \max_{f(w) \in \Omega_M^K} \int_{w \in \bar{W}^D} f(w) \mathrm{d}w. \tag{7}$$

In problem (7), the decision variables are the values of $f(w)$ over all possible $w \in \mathbb{R}^{N_W}$, thus there are infinitely many decision variables, and problem (7) is an infinite-dimensional linear programming. It maximizes the probability that w falls outside the dispatchable region W^D over all candidate PDFs, such that RTD will become infeasible. The optimal solution of problem (7) gives the worst-case PDF. However, it is difficult to solve (7) directly.

Associating a dual variable y_κ with each moment constraint in set Ω_M^K, we can derive the dual form of problem (7) in the spirit of the duality theory of conic linear programming [32] following the similar method in [26]

$$Z_D = \min_{y_\kappa} \sum_{\kappa \in J(K)} y_\kappa \sigma_\kappa \tag{8.1}$$

$$s.t.\ g(w) \geq 1, \forall w \in \bar{W}^D \tag{8.2}$$

$$g(w) \geq 0, \forall w \in E_B^W \tag{8.3}$$

where $g(w) = \sum_{\kappa \in J(K)} y_\kappa w^\kappa$ is a polynomial of w. Unlike the primal problem (7) that has infinite decision variables, the dual problem (8) has finite variables and an infinite number of constraints. In fact, we are optimizing objective (8.1) over the coefficients y_κ of the polynomial $g(w)$ that satisfies (8.2) and (8.3). We will leave the solution algorithm of problem (8) to the next section. It is clear that the optimum Z_P and Z_D depends on the current dispatch p, which influences the shape and size of $W^D/\bar{W}^D$.

Assumption 1: The moment vector $\bar{\sigma} = \{\sigma_\kappa\}, \forall \kappa \in J(K)$ is an interior point of the set of feasible moment vectors.

A moment vector is feasible if there exists a PDF such that $\sigma_\kappa = E[w_1^{k_1} w_2^{k_2} \cdots w_{N_W}^{k_{N_W}}], \forall \kappa \in J(K)$. Assumption 1 means all the moment vectors in a small neighborhood of $\bar{\sigma}$ are feasible. For more information about the feasibility of moment constraints, please consult [26] and [27]. Here we assume Assumption 1 always holds for the given data from an engineering perspective.

Proposition 1 [26], [27], [32]: If Assumption 1 is satisfied, then strong duality holds for problems (7) and (8), $Z_P = Z_D$.

III. Solution Approach

In this section, we develop tractable reformulations to estimate the optimum of dual problem (8). We will mainly focus on how to deal with constraint (8.2) and constraint (8.3), which actually restrict the coefficients of polynomial $g(w)$. To formulate these constraints in a tractable manner, we first introduce the notation of sum-of-squares (SOS) polynomial which is used to clarify non-negativity.

Definition 3: $g(w)$ is an SOS polynomial, if it can be represented as $g(w) = \sum_i h_i(w)^2$, where $h_i(w)$ are polynomials.

Let $\Sigma(w)$ be the set of all SOS polynomials. Clearly, if a polynomial $g(w) \in \Sigma(w)$, it must be nonnegative on $\mathbb{R}^{N_W}$.

A. SDP Reformulation Based Upper Bounding

1) Reformulation of Constraint (8.3): This constraint imposes non-negativity of $g(w)$ over the hypercube E_B^W, which is a subset of $\mathbb{R}^{N_W}$. We can adopt the paradigm of positivstellensatz. By introducing multiplier variables, we have the following proposition.

Proposition 2: $g(w) \geq 0$, $\forall w \in E_B^W$ holds if $\exists \lambda^l \geq 0, \lambda^u \geq 0$, such that

$$g(w) - l_E(w) \in \Sigma(w) \tag{9}$$

where polynomial $l_E(w) = w^T\lambda^l + (C^W - w)^T\lambda^u$.

This proposition is easy to understand. Denote by $g(w) - l_E(w) = p_0(w)$, for any $w \in E_B^W$, $0 \leq w \leq C_i^W, \forall i$ holds, so $l_E(w)$ must be non-negative as $\lambda^l \geq 0$, $\lambda^u \geq 0$. Moreover, $p_0(w) \in \Sigma(w) \geq 0$, thus $g(w) = p_0(w) + l_E(w) \geq 0$.

It is worth mentioning that since E_B^W is a semi-algebraic set, Proposition 2 is a special case of Putinar's Positivstellensatz [33]. Here we need to know the even order moments to guarantee the polynomial $g(w)$ has an even order. A polynomial with an odd order clearly does not belong to $\Sigma(w)$.

From the analysis above, constraint (8.3) can be approximated by the following SOS constraint

$$\begin{aligned} g(w) - l_E(w) = p_0(w) &\in \Sigma(w) \\ \lambda^l, \lambda^u &\geq 0. \end{aligned} \tag{10}$$

Clearly, constraint (10) is an sufficient condition of constraint (8.3), this approximation may lead to a larger optimum Z_D.

We emphasize that the variables in SOS constraint (10) are the coefficients y_κ of the polynomial $g(w)$ as well as the multipliers λ^l, λ^u. Variable w in polynomial $g(w)$ can be eliminated by introducing a positive semidefinite (PSD) matrix variable Q. To see this, let $\mathrm{w}^{(d)}$ be the vector of all monomials in variables $\{w_j\}, \forall j$ with highest degree d, for example, if $N_W = 2, d = 2$ then $\mathrm{w}^{(d)} = [1, w_1, w_2, w_1^2, w_1 w_2, w_2^2]^T$.

Proposition 3 [26]: Polynomial $p_0(w)$ of degree $2d$ is SOS if and only if $\exists Q \succeq 0$, such that $p_0(w) = \mathrm{w}^{(d)T} Q \mathrm{w}^{(d)}$.

In view of Proposition 3, we can write (10) as explicit PSD constraints in variable y_κ, Q and λ^l, λ^u by equating the coefficients of each monomial on both sides of (10). In this way, the enumeration of w over hypercube E_B^W in (8.3) is eliminated.

2) Reformulation of Constraint (8.2): Different from constraint (8.3), the non-dispatchable region $\bar{W}^D$ is not convex and the positivstellensats condition does not apply. Nevertheless, the dispatchable region W^D consists of linear inequalities, which allows us to represent $\bar{W}^D$ by using the union of half-spaces. So we can develop equivalent PSD constraints for constraint (8.2).

Recall (3), $W^D = \bigcap_i L_i^H$, $L_i^H = \{w \,|H_i w \leq h_i\}$, H_i is the i-th row of matrix H, h_i is the i-th element of vector h, the non-dispatchable region $\bar{W}^D = \bigcup_i \bar{L}_i^H$, the set $\bar{L}_i^H = \{w \,|H_i w \geq h_i\}$ is still a half-space (as mentioned before, we intentionally require $\bar{W}^D$ include the boundary of W^D, but this will not influence the probability as ∂W^D has zero measure on $\mathbb{R}^{N_W}$). Therefore, constraint (8.2) is equivalent to the following set of constraints

$$g(w) \geq 1, \forall w \in \bar{L}_i^H, \forall i, \tag{11}$$

For each half-space $\bar{L}_i^H$, we have a similar reformulation as that in Proposition 2. As a result, we arrive the following proposition.

Proposition 4: $g(w) \geq 1$, $\forall w \in \bar{W}^D$ if $\exists \rho = \{\rho_i\}, \forall i$ and $\rho_i \geq 0$, such that

$$g(w) - l_i^W(w) - 1 \in \Sigma(w), \forall i$$

where $l_i^W(w) = \rho_i(H_i w - h_i)$, ρ_i is the i-th element of ρ.

Similar to (10), constraint (8.2) can be approximated by

$$\begin{aligned} g(w) - l_i^W(w) - 1 \in \Sigma(w), \forall i \\ \rho_i \geq 0, \forall i. \end{aligned} \tag{12}$$

In view of Proposition 3, (12) is equivalent to a set of PSD constraints in variables y_κ, ρ and additional PSD matrix variables. The enumeration of w over set $\bar{W}^D$ in (8.2) is eliminated. For the same reason, constraint (12) is an sufficient condition of constraint (8.2). Replacing (8.2) with (12) may also lead to a larger optimum Z_D.

3) The Overall SDP: The previous outcomes allow us to reformulate the dual problem (8) using SOS constraints. The overall formulation is summarized as follows

$$\begin{aligned} Z_D^S = \min \sum_{\kappa \in J_K} y_\kappa \sigma_\kappa \\ s.t.\ \lambda^l \geq 0,\ \lambda^u \geq 0,\ \rho \geq 0 \\ g(w) - w^T\lambda^l - (C^W - w)^T\lambda^u \in \Sigma(w) \\ g(w) - \rho_i(H_i w - h_i) - 1 \in \Sigma(w), \forall i. \end{aligned} \tag{13}$$

Because SOS constraints imply PSD constraints, problem (13) is an SDP. In our implementation, SDP (13) is built using the SOS module in YALMIP [34], where SOS constraints are automatically transformed into equivalent PSD constraints.

Several discussions are provided below.

1) As mentioned before, Proposition 2 and Proposition 4 are only sufficient conditions for constraint (8.3) and constraint (8.2). In general, the optimal value Z_D^S of SDP (13) yields an upper bound of Z_D, even in the worst-case distribution.
2) Although we can solve SDP (13) and obtain the optimal solution y_κ, it is still difficult to recover the corresponding optimal solution of the primal problem (7), which means we do not have the worst-case distribution. Nevertheless, as long as the system operator knows the likelihood of infeasibility, the worst-case distribution usually becomes less important.

It should be mentioned that if only the first and second order moments are available, our problem can be addressed by the generalized Chebyshev's inequality in [25] and the generalized Gauss inequality in [28], given the dispatchable region W^D. We will briefly introduce these two methods in Appendix. The computation of W^D is efficient when N_W is not large [30].

B. Linear Programming Relaxation Based Lower Bounding

As problem (8) is a semi-infinite linear programming in variable y_κ, it is natural to use finite samples of $\bar{W}^D$ and E_B^W to construct the following linear programming relaxation

$$\begin{aligned} Z_D^L = \min \sum_{\kappa \in J_K} y_\kappa \sigma_\kappa \\ s.t.\ g(w) \geq 1,\ \forall w \in \bar{W}_R^D \\ g(w) \geq 0,\ \forall w \in E_R^W \end{aligned} \tag{14}$$

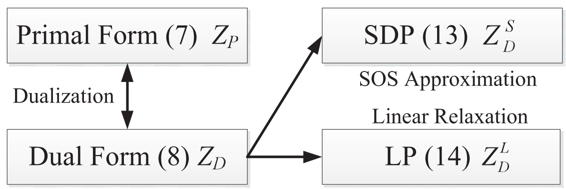

Fig. 2. Relationship of different methods.

where $\bar{W}_R^D$ and E_R^W is a finite subset of sets $\bar{W}^D$ and E_B^W, respectively, and the decision variable is y_κ. As the constraint of LP (14) is a necessary condition of constraints (8.2) and (8.3), the optimal value Z_D^L provides a lower bound of Z_D. Adding more elements into $\bar{W}_R^D$ and E_R^W can tighten the relaxation and reduce the gap.

Several discussions are provided as follows.

1) One possible way to generate sets $\bar{W}_R^D$ and E_R^W is to use the "grid points", which refers to the extreme points of smaller hypercubes E_i^S, such that $E_i^S \bigcap E_j^S = \emptyset, \forall i \neq j$ and $\bigcup_i E_i^S = E_B^W$. As the dispatchable region W^D is available, testing whether a grid point w_E^G belongs to W^D yields checking the validness of the linear inequality set $Hw_E^G \leq h$, which only involves algebraic operation.
2) The relationship among the primal problem (7), the dual problem (8), SDP (13) and LP (14) is shown in Fig. 2, their optimums satisfy $Z_D^S \geq Z_P = Z_D \geq Z_D^L$. Empirical results show that the gap $Z_D^S - Z_D^L$ tends to zero when more samples are used and the linear relaxation becomes tighter. This indicates $Z_D^S = Z_D$ thus SDP (13) which does not depend on the random samples gives good estimation of Z_D.
3) An important implication of the LP relaxation rises in the situation when the dispatchable region W^D is not available, thus the SDP reformulation, GCI and GGI are not applicable. Nevertheless, we can still build LP (14) and obtain a probability of infeasibility. In such circumstance, we need to solve an additional LP for each grid point w_E^G, in order to examine whether it belongs to W^D. In view of this, the computational efficiency may fail to satisfy the requirement of online application.
4) When the dimension of uncertainty grows, the number of grid points increases exponentially (the "curse of dimensionality"), so does the number of constraints in LP (14). Therefore, trade-off should be made between the number of sampled points and the precision of approximation.
5) In the case that the precision of grid points based approximation is not satisfactory, one way to enhance the LP relaxation is to incorporate a constraint violation checking and update sets $\bar{W}_R^D$ and E_R^W dynamically, yielding the following two steps

   Step 1: solve the master problem LP (14) with current $\bar{W}_R^D$ or E_R^W, the optimal solution is y_κ^*.

   Step 2: solve the following subproblems with $y_\kappa = y_\kappa^*$

$$g_1^* = \min_{w \in \bar{W}^D} g(w) \tag{15.1}$$

$$g_2^* = \min_{w \in E_B^W} g(w) \tag{15.2}$$

   and the optimal solution is w^*. If $g_1^* < 1$ or $g_2^* < 0$, the corresponding w^* is added into the set $\bar{W}_R^D$ or E_R^W, and then go to Step 1. The procedure continues until constraints (8.2) and (8.3) are met.

   In view of the complexity of $\bar{W}^D$, problem (15.1) can be replaced by a set of simpler problems

$$\begin{aligned} g_1^* &= \min\{g_{1i}^*, \forall i\} \\ g_{1i}^* &= \min_{w \in \bar{L}_i^H} g(w). \end{aligned} \tag{15.3}$$

   Because E_B^W is a hypercube and $\bar{L}_i^H$ is a hyperplane, the subproblems are polynomial optimizations with linear constraints. Interested readers are referred to [27] for a comprehensive discussion on polynomial optimization problems. Nevertheless, if a local optimum is acceptable, we can use any general nonlinear solver to find the solutions of subproblems. Because the constraint set is simple, the computation can be very efficient as long as the dimension of w is low, and this is just the situation of centralized wind power integration.

We end this section by providing further discussions on the data acquirement and computation related issues of the overall methodology.

1) Regarding the Moment Estimation: The moments are the only information required in the ambiguity set Ω_M^K. Compared with an accurate PDF, the moments can be directly computed from their mathematical definition using actual data, such as the recursive equations provided in [35], nevertheless, they can be estimated more comprehensively by using the methods in [36], [37] and references therein. Estimating the first and second order moments of stochastic wind generation has been well explored in power system applications. The first-order moment pertains to the wind power forecast, which has been extensively studied [38]; the second-order moment, or the covariance matrix of wind generation, is elaborated in [39]–[41]; general higher-order moments are investigated in [42]. In a probabilistic load flow study [43], higher order moments of wind power are simply calculated from their definitions. In this regard, we believe the moments are easier to obtain than the PDF.

2) Regarding the Sensitivity to the Errors of Moments: Although moment estimation is data-driven and relatively easy, computing higher-order moments may induce larger errors, as well as an ill-conditioned optimization problem. That is why we do not utilize moments with higher orders than 6. The comprehensive sensitivity analysis of input data perturbation can be found in [44]. As for our method, empirical results show that the worst-case probability is a continuous function of the moments. Nevertheless, if the point estimation of moment cannot meet desired accuracy, we can alternatively exploit the following ambiguity set with interval moment estimation

$$\Omega_M^K = \left\{ f(w) \left| \begin{array}{c} \int_{w \in E_B^W} w^\kappa f(w) \mathrm{d}w \leq \sigma_\kappa^m, \forall \kappa \in J(K) \\ \int_{w \in E_B^W} w^\kappa f(w) \mathrm{d}w \geq \sigma_\kappa^l, \forall \kappa \in J(K) \\ f(w) \geq 0, \ \forall w \in \mathbb{R}^{N_W} \end{array} \right. \right\}$$

where σ_κ^l and σ_κ^m is the lower bound and upper bound of k-th order moment σ_κ, $\sigma_0^m = \sigma_0^l = 1$, and then the corresponding dual problem will become

$$Z_D = \min_{y_\kappa} \sum_{\kappa \in J(K)} (y_\kappa^m \sigma_\kappa^m + y_\kappa^l \sigma_\kappa^l)$$
$$s.t.\ g(w) \geq 1,\ \forall w \in \bar{W}^D$$
$$g(w) \geq 0,\ \forall w \in E_B^W$$
$$y_\kappa^m \geq 0,\ y_\kappa^l \leq 0,\ \forall \kappa$$

where the polynomial $g(w) = \sum_{\kappa \in J(K)} (y_\kappa^m + y_\kappa^l) w^\kappa$. An SDP reformulation can be derived by using similar procedures proposed in this section.

3) Regarding the Computational Efficiency and Scalability: The proposed method relies on solving SDP, which is a convex optimization problem and admits efficient computation using commercial softwares, however, difficulty arises when the number of wind farms grows and the orders of moments become high. This is because in Proposition 3, the dimension of vector $\mathrm{w}^{(d)}$ is $C_d^{N_W+d}$, where C_m^n is the combination number of choosing m items from n items, and $d = K/2$ is the highest degree of monomials in $\mathrm{w}^{(d)}$, so the dimension of the PSD matrix Q will be $C_d^{N_W+d} \times C_d^{N_W+d}$, which grows quickly with N_W and d increasing, but is independent of the power system model. In the case study, we demonstrate that moments up to 4-th order are enough to procure a reasonable result. As for the GCI and GGI methods, the complexity depends on the number of constraints in W^D, which also grows quickly with N_W increasing. Moreover, computing W^D requires solving MILP repeatedly, which is less efficient when N_W is large. W^D is readily accessible only when the linear redispatch policy is adopted in RTD [31]. In this regard, all methods discussed in this paper are restricted to the instances of bulk power systems with several large renewable generation centers. Computation time on a moderately sized power system with up to 7 wind farms will be illustrated in case study.

4) Regarding the Numerical Issue: Because the moments of different orders exhibit different orders of magnitudes, problem (8) may be ill-conditioned and it is important to reduce the conditional number of SDP (13) from computational perspective. One possible way to alleviate this difficulty would be to use the "unit value", that is to divide the output data by a constant, called the base value. We can choose different base values for different wind farms. Consequently, we should also use the formulation of W^D with the coordinates represented by the unit value, i.e.,

$$W^D = \{\bar{w} \in \mathbb{R}^{N_W} | H_u \bar{w} \leq h_u\}$$

with $H_u = HD(w^B)$, $h_u = h$, where H and h are the coefficient matrices in (3), $D(w^B)$ is a diagonal matrix whose non-zero elements are the base values of wind farms. One possible choice of the base value may be $w^B = 0.5C^W$. Another way is to use the center moment. Similarly, we should also represent W^D within the coordinates whose origin is located at w^e

$$W^D = \{\Delta w \in \mathbb{R}^{N_W} | H_c \Delta w \leq h_c\}$$

with $H_c = H$, $h_c = h - Hw^e$, where w^e is the forecast output of wind farms. Empirical studies suggest that the former one is usually more effective.

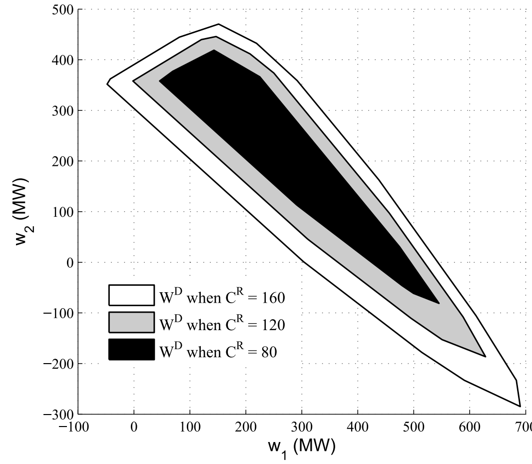

Fig. 3. Dispatchable region under different available costs.

IV. Case Studies

In this section, the proposed method is applied to the IEEE 118-bus system and compared with GCI and GGI to verify their performances in practical applications. System data are provided at: http://motor.ece.iit.edu/data/JEAS_IEEE118.doc. TED (5) is used to compute the current generation strategy p^f. All numeric experiments are conducted on a laptop with Intel i5-3210M CPU and 4 GB memory. SDPs and LPs are solved by MOSEK [45].

The test system possesses 54 conventional generators and 186 transmission lines. In the considered dispatch interval, the total demand is 5000 MW. According to the setting in [11], the regulation cost coefficient d_i^+/d_i^- of each generator is set to be 10% of its production cost coefficient b_i. The ramping limit R_i^+/R_i^- is assumed to be 25% of its maximal output P_i^u. To illustrate the dispatchable region W^D clearly, we consider two wind farms with $C_1^W = C_2^W = 500$ MW and $w_1^e = w_2^e = 250$ MW in this case study. They are connected to the system at bus #70 (Area 1) and bus #100 (Area 3). The impact of the forecast accuracy and the available cost C^R on the probability of operating feasibility is investigated.

According to [38], the root mean-square error of the hourly-ahead wind generation forecast is around 10% of its prediction. In our experiments, we change the square root variance σ_j of forecast error from 8% to 12% of w_j^e. In this section, SDP (13) is called SDP-X method for short, where X is the available order of moments. The optimal value provides an upper bound for the concerning probability of infeasible RTD. Similarly, LP (14) is called LPR-X for short, whose optimal value provides a lower bound for the probability of infeasibility. Details about GCI and GGI can be found in Appendix. The polyhedral form (3) of W^D under different cost C^R is computed by using the method in [30] and illustrated in Fig. 3. Clearly, W^D grows larger with C^R increasing, indicating higher flexibility of RTD.

The computation time of SDP method typically varies from 1–2 seconds. As for LPR method, we divide E_B^W into 400 sub-rectangles to retrieve the grid points. The computation time is around 2 seconds. The computation time of GCI and GGI method is around 0.8 second. Detailed results are shown through Figs. 4 – 6. The outcomes offered by SDP method and

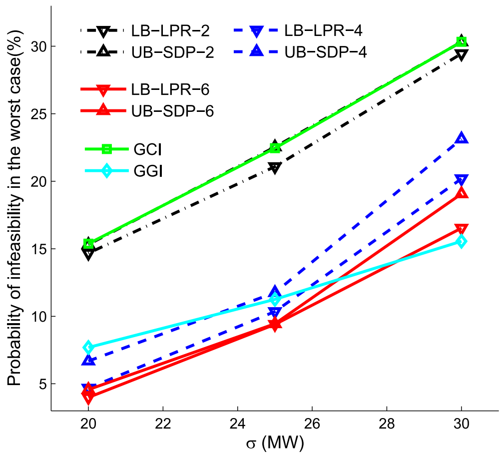

Fig. 4. Probability of infeasibility when $C^R = 80$ MBtu.

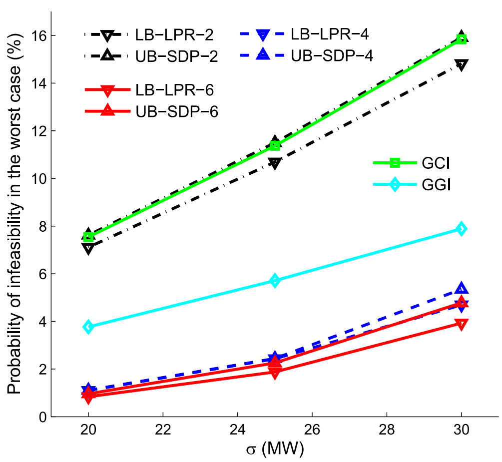

Fig. 5. Probability of infeasibility when $C^R = 120$ MBtu.

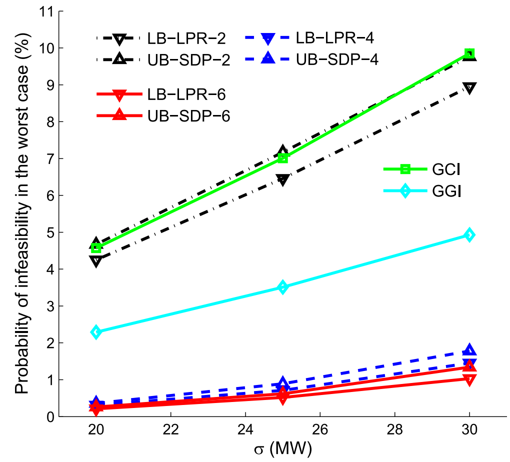

Fig. 6. Probability of infeasibility when $C^R = 160$ MBtu.

TABLE I
PROBABILITY OF INFEASIBILITY UNDER GAUSSIAN DISTRIBUTION

Square root of variance	Probability of infeasibility (%) $C^R = 80$	$C^R = 120$	$C^R = 160$
$\sigma = 20$MW	0.81	0.01	0.00
$\sigma = 25$MW	2.69	0.21	0.00
$\sigma = 30$MW	6.44	0.82	0.08

LPR method with the same X is plotted using the same color. The gap between the upper bound and lower bound shown in Figs. 4–6 is acceptable, indicates both methods provide good approximations for problem (8). Clearly, with the available RTD cost C^R increasing, the probability of infeasibility decreases because W^D grows larger. For SDP method, when higher order moments are used, the conservativeness will be reduced. For LPR method, we find if more grid points are used in LP (14), the gap $Z_D^S - Z_D^L$ can be further reduced and eventually tends to 0. This also indicates that the probability offered by SDP method is exact. It is worth mentioning that incorporating higher order moments will increase the conditional number and problem size, thus challenging the computation, we do not recommend using moments with higher order than 6. By noticing that the results of SDP-6 is only slightly improved compared with that of SDP-4, the latter may be the best choice in practice.

These figures also indicate that SDP-2 provides almost the same results as the GCI method, because they actually yield very similar problem. By considering the unimodality of wind power distribution, the GGI method remarkably reduces the conservativeness of the GCI method. It's also interesting to note that the probability offered by GGI is almost half of that offered by GCI, coinciding with the univariate situation, where the former is 4/9 of the latter. It is also found that SDP-4 and SDP-6 provide a lower probability than GGI in most cases except when C^R is small (W^D is small) and the forecast is less accurate (the variance is large).

Please keep in mind that the probabilities offered by all these methods correspond to the worst wind power distribution. To demonstrate the probability of infeasibility under a specified distribution, say Gaussian distribution, we carry out Monte Carlo simulation (MCS) which proceeds as follows: 1) generate 10,000 wind generation scenarios from the PDF of Gaussian distribution $N(w^e, \Theta)$, the detected number of non-feasible scenarios $N = 0$; 2) for each scenario w, if $w \notin W^D$, $N = N + 1$. The probability of infeasible RTD provided by MCS is $N/10^4$. Results are shown in Table I, from which we can see, the infeasible probability under Gauss distribution is smaller than those in Figs. 4 – 6, because Gaussian distribution is only a candidate PDF in set Ω_M^K rather than the worst one, MCS only yields a lower bound of Z_P. The actual infeasible probability may be either lower or higher than the result of MCS, depending on the actual PDF of wind power, but must be smaller than the optimal value of problem (8). One can certainly use other PDFs to carry out MCS. However, as mentioned before, any specific PDF may not completely fit the actual data. Since the time scale of RTD is short, the Gaussian distribution is eligible for our study [5]–[7].

Finally, we test the computation time of each method with larger dimensional uncertainty. We change the number of wind farms N_W in the system. To maintain a proper penetration level of renewable energy, we keep the total wind power generation at 1000 MW, indicating $w_j^e = 1000/N_W$, and choose $\sigma_j = 0.1 w_j^e$ for each wind farm. The computation times of SDP method

TABLE II
COMPUTATION TIME (S)

	$N_W = 4$	$N_W = 5$	$N_W = 6$	$N_W = 7$
SDP-2	2.42	2.97	6.39	20.2
SDP-4	2.95	4.28	15.4	97.6
SDP-6	8.01	41.6	179	*
GGI	1.89	3.01	11.2	105
GCI	1.64	2.27	7.21	63.5

[SDP (13) in Section III], GCI [SDP (A2) in the Appendix] and GGI [SDP (A4) in the Appendix] are shown in Table II.

Table II demonstrates that the computation time of all methods increases quickly when $N_W = 7$, and SDP-6 fails to return a solution. Other methods can solve the problem in reasonable time and satisfy the requirement of online application. The reason is, the complexity of SDP (13) mainly depends on N_W, as analyzed in Section III, while the complexity of SDP (A2) and SDP (A4) mainly depends on the number of constraints in W^D. When the number of wind farms increases, both the dimension of PSD matrix variable in SDP (13) and the number of constraints in W^D grow quickly. It is also worth mentioning that when N_W grows larger, computing W^D will become more challenging. This difficulty can be partly alleviated by assuming the redispatch is a linear function of the wind power forecast error, in such circumstance, the coefficients H and h of the dispatchable region is explicitly given (Proposition 1, [31]). Taking the computational efficiency and reliability, solution conservativeness, as well as implementation issues into account, SDP-4 might be the best choice in practical usage.

V. CONCLUSION

This paper discusses how to estimate the probability of an infeasible RTD in the absence of exact PDFs of volatile wind generations. The proposed method can provide the operator intuitive information on the impact of uncertain renewable generation on the system dispatch, and a quantitative measure on how reliable the power system is. Numerical experiments on the IEEE 118-bus system demonstrate the applicability of our method on moderately sized power systems. From the aspect of computational efficiency, our method is suitable for bulk power systems with centralized renewable integration. From the aspect of data acquisition, when only the first and second order moments are available, we recommend the GGI method; when higher-order moments are provided, we recommend comparing the probabilities offered by SDP-4 and the GGI method, and trust the smaller one. The experimental results also suggest that the proposed method is inevitably conservative to some extent, as the worst-case distribution is considered. However, it is still desired by operators due to its computational advantages and independence on the exact PDFs. One possible way to reduce the conservativeness is to incorporate the unimodality of the PDFs of wind generation in the high-order SDP method, which is our ongoing research. Notice that most of our discussion is generic and could be applied to many problems that are modeled by bounding the probability of uncertain events outside a polytope, we believe this method should be useful in a broader class of power system applications, where the operator needs to quantify the impact of uncertainty on system reliability, such as power system planning and policy making.

APPENDIX

We will define some notations in a more convenient way for GCI and GGI. The dispatchable region in the coordinate of forecast error Δw is expressed as

$$W^D = \{\Delta w | a_i^T \Delta w \le b_i,\ i = 1, \ldots, k\}. \tag{A1}$$

The first-order moment of Δw is 0, and the second-order moment of Δw is the covariance matrix Θ. Recall (3), notice $w = w^e + \Delta w$, we have

$$W^D = \{\Delta w \in \mathbb{R}^{N_W} | H(w^e + \Delta w) \le h\}.$$

Therefore, a_i and b_i in equation (A1) can be calculated as

$$a_i^T = H_i, \quad b_i = h_i - H_i w^e$$

where H_i is the i-th row of matrix H.

Generalized Chebyshev Inequality Approach: GCI in [25] extends the traditional Chebyshev inequality to a multivariate setting and provides the probability of a random vector falling outside polytope (A1) under the worst distribution in Ω_C described by

$$\Omega_C = \left\{ f(\Delta w) \left| \begin{array}{c} \int_{\mathbb{R}^{N_W}} f(\Delta w) \mathrm{d}w = 1, f(\Delta w) \ge 0 \\ \int_{\mathbb{R}^{N_W}} \Delta w f(\Delta w) \mathrm{d}w = 0 \\ \int_{\mathbb{R}^{N_W}} \Delta w \Delta w^T f(\Delta w) \mathrm{d}w = \Theta \end{array} \right. \right\}.$$

GCI can boil down to a single SDP [25]

$$\begin{aligned} \sup_{f(w) \in \Omega_C} \Pr[w \notin W^D] = \max \sum_{i=1}^{k} \lambda_i \\ s.t.\ z_i \in \mathbb{R}^{N_W}, Z_i \in \mathbb{S}^{N_W}, \forall i = 1, \ldots, k \\ \lambda_i \in \mathbb{R},\ a_i^T z_i \ge b_i \lambda_i,\ \forall i = 1, \ldots, k \\ \begin{pmatrix} Z_i & z_i \\ z_i^T & \lambda_i \end{pmatrix} \succeq 0,\ \forall i = 1, \ldots, k \\ \begin{pmatrix} \Theta & 0 \\ 0 & 1 \end{pmatrix} \succeq \sum_{i=1}^{k} \begin{pmatrix} Z_i & z_i \\ z_i^T & \lambda_i \end{pmatrix}. \end{aligned} \tag{A2}$$

However, this method often appears to be very pessimistic. The reason is that the worst-case distribution has only few discretization points [25], instead of a continuous PDF.

Generalized Gauss Inequality Approach: To alleviate the conservativeness of the GCI approach without sacrificing the convexity, [28] proposes the GGI approach by exploring the unimodality which excludes the discrete distributions from the candidate PDFs, thus the conservatism can be reduced remarkably. Intuitively, the PDF of a unimodal distribution has only one maximum, in other words, the PDF is non-increasing when leaving from the maximum. For rigorous mathematical definition of unimodality, please refer to [28]. Although the PDF of wind speed over a long time period may be multi-modal, we believe the PDF of wind power should be unimodal around its prediction in the time frame of RTD. Meanwhile, we do not require the PDF should be symmetric.

For the univariate case, Gauss proved that the Chebyshev inequality can be improved as follows when the PDF is known to be unimodal [28]

$$\Pr[|\xi-\mu|\geq k\sigma]\leq\begin{cases}\frac{4}{9k^2}, & \text{if } k>\frac{2}{\sqrt{3}};\\ 1-\frac{k}{\sqrt{3}}, & \text{otherwise;}\end{cases} \tag{A3}$$

When Gauss inequality (A3) is applied to the simplest case in Introduction, the probability will typically be reduced to 4/9 of that provided by Chebyshev inequality.

The work in [28] generalizes Gauss's result to incorporate multiple random variables. The candidate PDFs in GGI belongs to

$$\Omega_G=\left\{f(\Delta w)\,\middle|\,\begin{matrix}f(\Delta w)\in\Omega_C\\ f(\Delta w)\text{ is unimodal}\end{matrix}\right\}.$$

GGI still comes down to a single SDP [28]

$$\begin{aligned}
\sup_{f(w)\in\Omega_G}\ \Pr[w\notin W^D]&=\max\sum_{i=1}^{k}(\lambda_i-t_{i,0})\\
s.t.\ z_i\in\mathbb{R}^{N_W},&\,Z_i\in\mathbb{S}^{N_W},\lambda_i\in\mathbb{R},\forall i=1,\ldots,k\\
t_i\in\mathbb{R}^{l+1},&\,l=\lceil\log_2 N_W\rceil,\forall i=1,\ldots,k\\
\begin{pmatrix}Z_i & z_i\\ z_i^T & \lambda_i\end{pmatrix}&\succeq 0,\ a_i^Tz_i\geq 0,\ t_i\geq 0,\forall i=1,\ldots,k\\
\begin{pmatrix}\frac{N_W+2}{N_W}\Theta & 0\\ 0 & 1\end{pmatrix}&\succeq\sum_{i=1}^{k}\begin{pmatrix}Z_i & z_i\\ z_i^T & \lambda_i\end{pmatrix}\\
\left\|\begin{matrix}2\lambda_i b_i\\ t_{i,l}b_i-a_i^Tz_i\end{matrix}\right\|_2&\leq t_{i,l}b_i-a_i^Tz_i,\ \forall i=1,\ldots,k\\
\left\|\begin{matrix}2t_{i,j+1}\\ t_{i,j}-\lambda_i\end{matrix}\right\|_2&\leq t_{i,j}+\lambda_i,\forall j\in EV,\forall i=1,\ldots,k\\
\left\|\begin{matrix}2t_{i,j+1}\\ t_{i,j}-t_{i,l}\end{matrix}\right\|_2&\leq t_{i,j}+t_{i,l},\forall j\in OD,\forall i=1,\ldots,k\\
EV&=\left\{j\in\{0,\ldots,l-1\}:\left\lceil\frac{N_W}{2^j}\right\rceil\text{ is even}\right\}\\
OD&=\left\{j\in\{0,\ldots,l-1\}:\left\lceil\frac{N_W}{2^j}\right\rceil\text{ is odd}\right\}
\end{aligned} \tag{A4}$$

where $\lceil\ \rceil$ stands for the round function towards plus infinity.

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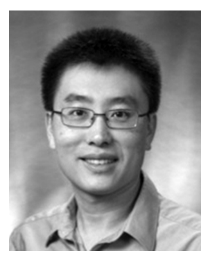

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