53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Stability constrained incentive design for distributed frequency control of power grid

Minghui Zhu and Na Li

Abstract— **In this paper, we consider the problem of stability constrained incentive design for distributed frequency control of multi-machine power systems. We aim to design incentive mechanisms under which control authorities are incentivized to implement distributed controllers which can ensure frequency stability. To solve the problem, we propose a reward based mechanism and determine a lower bound on the rewards such that the induced Nash equilibrium realizes the above objective on dynamic behavior of power grid.**

I. INTRODUCTION

In power grid, a fundamental objective is to reliably balance power generations and demands. The objective has been facilitated by the active engagement of end users, the high penetration of renewable energy and the wide deployment of advanced sensing, communication and control. Meanwhile, these new components present new challenges to grid operation. In particular, distributed energy resources are managed by heterogeneous control authorities who seek for different and even partially conflicting subobjectives. In order to ensure grid functions and performance, the system operator may want to influence the preferences of control authorities such that the gaps between social welfare; e.g., grid stability, and self interests are reduced. This task is challenged by the fact that the owners of distributed energy resources may not be unwilling to disclose certain private information.

Contributions. In this paper, we study incentive design for distributed frequency control of multi-machine power systems. To reduce the disclosure of private information of control authorities, we convert a dynamic mechanism design into a static parametric non-cooperative game. In particular, we first formulate the problem of stability constrained incentive design where the system operator incentivizes control authorities to choose distributed controllers which induce proper static gain functions and further ensure frequency stability. Inspired by fixed-prize raffles in [13], we propose a reward based incentive mechanism where each control authority obtains a portion of a fixed reward and its reward is proportional to its contribution to grid stability. The contribution of each control authority is characterized by the inverse of its linear gain function. The incentive mechanism presents a non-cooperative game among control authorities. Once receiving the rewards corresponding to a Nash equilibrium, control authorities choose distributed controllers to commit to their proposals. We identify the lower bound on the reward under which the induced Nash equilibrium can enforce the stability condition. We also provide other analytic properties of the incentive mechanism.

Literature review. Our work is related to algorithmic (or dynamic) mechanism design [14], [16], [17], [20]. In algorithmic mechanism design, individual agents follow dynamic algorithms or controllers which are specified *a priori* by a system operator. The faithful implementation is achieved by that the system operator provides side payment mechanisms to change the utilities of individual agents. In order to compute desired algorithms or controllers, the system operator needs to access the structures and parameters of the dynamic systems of individual agents. In addition, the computation of side payments requires the system operator to know the utilities of individual agents. However, the above information could be private for individual agents and thus may not be accessible to the system operator. Compared with algorithmic mechanism design [14], [16], [17], [20], our scheme is privacy aware. In particular, the system operator only needs to know the gain functions of control authorities, and distributed controllers are designed by individual control authorities instead of the system operator.

In the literature of micro economics, there are a large number of incentive or pricing schemes which can partially mitigate selfish behavior in competitive scenarios. Theses schemes have been applied to many domains, including the demand response of power grid [1], [4], [5], [21], communication networks [7] and transportation networks [12]. However, the set of papers do not consider dynamic systems.

Distributed control of the power grid has received substantial attention. The classic distributed control includes power system stabilizer (PSS) and automatic generation control (AGC) [6], [9], [10], [22], [23]. These papers do not consider selfish behavior of individual controllers.

Privacy of information systems has been extensively studied in; e.g., [2], [3], [18]. Recently, privacy of cyber-physical systems has been attracting increasing attention especially in the smart grid [11], [15], [19], [24]. While privacy of algorithmic mechanism design has not been investigated.

Notations. Denote the supremum norm of the truncation of $u(t)$ in $[t_1, t_2]$ by $\|u\|_{[t_1,t_2]} \triangleq \sup_{t_1 \le t \le t_2} \|u(t)\|$. The dynamic system $\dot{x} = f(x, u, t)$ is called input-to-state stable if there exist class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and class $\mathcal{K}$ functions $\gamma(\cdot)$ such that for all $x(t_0) \in \mathbb{R}^n$ the following holds for all $t \ge t_0$: $\|x(t)\| \le \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u\|_{[t_0,t]})\}$, where the functions of β and γ are time independent. γ is referred to as the gain function.

M. Zhu is with the Department of Electrical Engineering, Pennsylvania State University, 201 Old Main, University Park, PA, 16802, muz16@psu.edu. N. Li is with the Department of Electrical Engineering, Harvard University, 29 Oxford Street, Cambridge, MA 02138, na_li@mit.edu.

978-1-4673-6090-6/14/$31.00 ©2014 IEEE

II. PRELIMINARY ON DISTRIBUTED FREQUENCY CONTROL

In this section, we will investigate distributed frequency control of interconnected synchronous generators. The parameters used in this section are summarized in Table I.

TABLE I: Generator variables

w	angular frequency
θ	phase angle
P_M	mechanical power
P_M^*	set-point of mechanical power
P_L	actual load
P_L^*	forecasted load
P_v	stream valve position
P_v^*	set-point of stream valve position
P_{ij}	tie-line power flow between control authorities i and j
P_{ref}	reference power
P_{ref}^*	set-point of reference power
v	wind speed
P_w	wind power

A. Power system model

We consider a power system comprised by a collection of interconnected buses. Each bus represents a control authority which may consist of a variety of generators and/or loads. It has been a common practice to lump all the generators (resp. loads) of a control authority as a single generator (resp. load). At each control authority, we assume that there is a mechanical generation P_{M_i} and load P_{L_i}. The control authorities are connected with each other through the power flow P_{ij}.

1) Generator dynamics: We will adopt the model of synchronous generators in [10] for each control authority. The notation Δ is used to indicate a deviation from nominal value $[\theta_i^*, w_i^*, P_{M_i}^*, P_{v_i}^*, P_{ref_i}^*, P_{L_i}^*]^T$. For example, Δw_i represents the deviation of angular frequency from the constant set-point w^*; e.g., 60 Hz.

At time instant t, the state-space model of control authority i is given by the following:

$$\begin{aligned}\frac{d\Delta\theta_i}{dt} &= 2\pi\Delta w_i,\\ \frac{d\Delta w_i}{dt} &= -\frac{1}{M_i}\big(D_i\Delta w_i + \sum_{j\in\mathcal{N}_i}\Delta P_{ij}\\ &\quad - \Delta P_{M_i} + \Delta P_{L_i}\big),\\ \frac{d\Delta P_{M_i}}{dt} &= -\frac{1}{T_{CH_i}}\big(\Delta P_{M_i} - \Delta P_{v_i}\big),\\ \frac{d\Delta P_{v_i}}{dt} &= -\frac{1}{T_{G_i}}\big(\Delta P_{v_i} + \frac{1}{R_i}\Delta w_i - \Delta P_{ref_i}\big), \qquad (1)\end{aligned}$$

where $\mathcal{N}_i \subseteq V\setminus\{i\}$ represents the set of neighboring control authorities. In (1), the power flow between control authorities i and j is modeled by:

$$\Delta P_{ij}(t) = T_{ij}(\Delta\theta_i(t) - \Delta\theta_j(t)). \qquad (2)$$

The swing dynamics, the second equation in (1), captures the frequency evolution. The first equation in (1) and (2) describe the branch flow dynamics. The third and fourth equations in (1) stand for the dynamic system of turbine-governor which is controlled via the reference point ΔP_{ref_i}.

For notational simplicity, we drop Δ in the states and let $P_{L_i} = 0$. Denote the new input $u_i = \frac{1}{R_i}w_i - P_{ref_i}$.

B. Distributed frequency control

Inspired by backstepping [8], we will derive that the gain function of each control authority can be rendered arbitrarily small.

1) Coordination transformation one: We define the first coordinate transformation as follows:

$$\begin{aligned}&w_i^* = -\frac{1}{2\pi}k_{i,1}\theta_i, \quad \hat{\theta}_i = \theta_i, \quad \hat{P}_{M_i} = P_{M_i},\\ &\hat{P}_{v_i} = P_{v_i}, \quad \hat{u}_i = u_i, \quad \hat{w}_i = w_i - w_i^*,\end{aligned}$$

and have the following error dynamics:

$$\begin{aligned}\frac{d\hat{\theta}_i}{dt} &= -k_{i,1}\hat{\theta}_i + 2\pi\hat{w}_i,\\ \frac{d\hat{w}_i}{dt} &= -\frac{1}{M_i}\big(D_i\hat{w}_i + D_i w_i^* + \sum_{j\in\mathcal{N}_i}P_{ij} - \hat{P}_{M_i}\big) - \dot{w}_i^*,\\ \frac{d\hat{P}_{M_i}}{dt} &= -\frac{1}{T_{CH_i}}\big(\hat{P}_{M_i} - \hat{P}_{v_i}\big),\\ \frac{d\hat{P}_{v_i}}{dt} &= -\frac{1}{T_{G_i}}\big(\hat{P}_{v_i} + u_i\big),\end{aligned}$$

where $\dot{w}_i^* = \frac{1}{2\pi}k_{i,1}^2\hat{\theta}_i - k_{i,1}\hat{w}_i$.

2) Coordination transformation two: We define the second coordinate transformation as follows:

$$\begin{aligned}\hat{P}_{M_i}^* &= M_i(-k_{i,2}\hat{w}_i + \dot{w}_i^*) + D_i\hat{w}_i + D_i w_i^* + \sum_{j\in\mathcal{N}_i}T_{ij}\hat{\theta}_i\\ &= a_i(k_{i,1},k_{i,2})\hat{\theta}_i + b_i(k_{i,1},k_{i,2})\hat{w}_i\\ \bar{\theta}_i &= \hat{\theta}_i, \quad \bar{w}_i = \hat{w}_i, \quad \bar{P}_{M_i} = \hat{P}_{M_i} - \hat{P}_{M_i}^*, \quad \bar{P}_{v_i} = \hat{P}_{v_i},\end{aligned}$$

and have the following error dynamics:

$$\begin{aligned}\frac{d\bar{\theta}_i}{dt} &= -k_{i,1}\bar{\theta}_i + 2\pi\bar{w}_i,\\ \frac{d\bar{w}_i}{dt} &= -k_{i,2}\bar{w}_i + \frac{1}{M_i}\big(\sum_{j\in\mathcal{N}_i}T_{ij}\bar{\theta}_j + \bar{P}_{M_i}\big),\\ \frac{d\bar{P}_{M_i}}{dt} &= -\frac{1}{T_{CH_i}}\big(\bar{P}_{M_i} + \hat{P}_{M_i}^* - \bar{P}_{v_i}\big) - \dot{\hat{P}}_{M_i}^*,\\ \frac{d\bar{P}_{v_i}}{dt} &= -\frac{1}{T_{G_i}}\big(\bar{P}_{v_i} + u_i\big),\end{aligned}$$

where

$$\begin{aligned}\dot{\hat{P}}_{M_i}^* &= c_i(k_{i,1},k_{i,2})\hat{\theta}_i + d_i(k_{i,1},k_{i,2})\hat{w}_i(t)\\ &\quad + e_i(k_{i,1},k_{i,2})\hat{P}_{M_i} + \frac{b_i(k_{i,1},k_{i,2})}{M_i}\sum_{j\in\mathcal{N}_i}T_{ij}\hat{\theta}_j\end{aligned}$$

In $\dot{\hat{P}}_{M_i}^*$, the term of $\frac{b_i(k_{i,1},k_{i,2})}{M_i}\sum_{j\in\mathcal{N}_i}T_{ij}\hat{\theta}_j$ is not accessible to control authority i due to the lack of communication.

3) Coordination transformation three: We define the third coordinate transformation as follows:

$$\begin{aligned}\bar{P}_{v_i}^* &= -k_{i,3}T_{CH_i}\bar{P}_{M_i} + \bar{P}_{M_i} + \hat{P}_{M_i}^* \\ &\quad + T_{CH_i}(c_i\hat{\theta}_i + d_i\hat{w}_i(t) + e_i\hat{P}_{M_i}) \\ &= a_i'(k_{i,1},k_{i,2},k_{i,3})\hat{\theta}_i + b_i'(k_{i,1},k_{i,2},k_{i,3})\hat{w}_i(t) \\ &\quad + c_i'(k_{i,1},k_{i,2},k_{i,3})\bar{P}_{M_i}(t) \\ \tilde{\theta}_i &= \bar{\theta}_i, \quad \tilde{w}_i = \bar{w}_i, \\ \tilde{P}_{M_i} &= \bar{P}_{M_i}, \tilde{P}_{v_i} = \bar{P}_{v_i} - \bar{P}_{v_i}^*\end{aligned}$$

and have the following error dynamics:

$$\begin{aligned}\frac{d\tilde{\theta}_i}{dt} &= -k_{i,1}\tilde{\theta}_i + 2\pi\tilde{w}_i, \\ \frac{d\tilde{w}_i}{dt} &= -k_{i,2}\tilde{w}_i + \frac{1}{M_i}\big(\sum_{j\in\mathcal{N}_i} T_{ij}\tilde{\theta}_j + \tilde{P}_{M_i}\big), \\ \frac{d\tilde{P}_{M_i}}{dt} &= -k_{i,3}\tilde{P}_{M_i} + \frac{1}{T_{CH_i}}\tilde{P}_{v_i} \\ &\quad + \frac{b_i(k_{i,1},k_{i,2})}{M_i}\sum_{j\in\mathcal{N}_i} T_{ij}\tilde{\theta}_j, \\ \frac{d\tilde{P}_{v_i}}{dt} &= -\frac{1}{T_{G_i}}\big((\tilde{P}_{v_i} + \bar{P}_{v_i}^*) + u_i\big) - \dot{\bar{P}}_{v_i}^*,\end{aligned}$$

where

$$\begin{aligned}\dot{\hat{P}}_{v_i}^* &= c_i'(k_{i,1},k_{i,2},k_{i,3})\hat{\theta}_i + d_i'(k_{i,1},k_{i,2},k_{i,3})\hat{w}_i(t) \\ &\quad + e_i'(k_{i,1},k_{i,2},k_{i,3})\hat{P}_{M_i} + \frac{b_i'(k_{i,1},k_{i,2},k_{i,3})}{M_i}\sum_{j\in\mathcal{N}_i} T_{ij}\hat{\theta}_j\end{aligned}$$

4) Distributed controller synthesis: Choose u_i such that $-\frac{1}{T_{G_i}}\big((\tilde{P}_{v_i}+\bar{P}_{v_i}^*)+u_i\big) - (c_i'\hat{\theta}_i + d_i'\hat{w}_i + e_i'\hat{P}_{M_i}) = -k_{i,4}\tilde{P}_{v_i}$ and have the following error dynamics:

$$\begin{aligned}\frac{d\tilde{\theta}_i}{dt} &= -k_{i,1}\tilde{\theta}_i + 2\pi\tilde{w}_i, \\ \frac{d\tilde{w}_i}{dt} &= -k_{i,2}\tilde{w}_i + \frac{1}{M_i}\big(\sum_{j\in\mathcal{N}_i} T_{ij}\tilde{\theta}_j - \tilde{P}_{M_i}\big), \\ \frac{d\tilde{P}_{M_i}}{dt} &= -k_{i,3}\tilde{P}_{M_i} + \frac{1}{T_{CH_i}}\tilde{P}_{v_i} \\ &\quad + \frac{b_i(k_{i,1},k_{i,2})}{M_i}\sum_{j\in\mathcal{N}_i} T_{ij}\tilde{\theta}_j, \\ \frac{d\tilde{P}_{v_i}}{dt} &= -k_{i,4}\tilde{P}_{v_i} - \frac{b_i'(k_{i,1},k_{i,2},k_{i,3})}{M_i}\sum_{j\in\mathcal{N}_i} T_{ij}\tilde{\theta}_j.\end{aligned}$$

5) Stability analysis: Let $\tilde{x}_i \triangleq [\tilde{\theta}_i \;\; \tilde{w}_i \;\; \tilde{P}_{M_i} \;\; \tilde{P}_{v_i}]^T$. Choose $V_i(\tilde{x}_i) = \frac{1}{2}\|\tilde{x}_i\|^2$. So we have

$$\begin{aligned}\dot{V}_i &= -k_{i,1}\tilde{\theta}_i^2 + 2\pi\tilde{\theta}_i\tilde{w}_i \\ &\quad - k_{i,2}\tilde{w}_i^2 + \frac{1}{M_i}\big(\sum_{j\in\mathcal{N}_i} T_{ij}\tilde{w}_i\tilde{\theta}_j + \tilde{w}_i\tilde{P}_{M_i}\big) \\ &\quad - k_{i,3}\tilde{P}_{M_i}^2 + \frac{1}{T_{CH_i}}\tilde{P}_{M_i}\tilde{P}_{v_i} \\ &\quad + \frac{b_i(k_{i,1},k_{i,2})}{M_i}\sum_{j\in\mathcal{N}_i} T_{ij}\tilde{P}_{M_i}\tilde{\theta}_j - k_{i,4}\tilde{P}_{v_i}^2 \\ &\quad + \frac{b_i'(k_{i,1},k_{i,2},k_{i,3})}{M_i}\sum_{j\in\mathcal{N}_i} T_{ij}\tilde{P}_{v_i}\tilde{\theta}_j.\end{aligned}$$

Note that $T_{ij}b_i(k_{i,1},k_{i,2})\tilde{P}_{M_i}\tilde{\theta}_j \leq \frac{1}{2}((T_{ij}b_i(k_{i,1},k_{i,2}))^2\tilde{P}_{M_i}^2 + \tilde{\theta}_j^2)$ and $T_{ij}b_i'(k_{i,1},k_{i,2},k_{i,3})\tilde{P}_{v_i}\tilde{\theta}_j \leq \frac{1}{2}(T_{ij}b_i'(k_{i,1},k_{i,2},k_{i,3})^2\tilde{P}_{v_i}^2 + \tilde{\theta}_j^2)$. So $\tilde{x}_i$ is input-to-state stable with respect to $\tilde{x}_j$ with a linear gain

$$\begin{aligned}\gamma_i &= |\mathcal{N}_i|/\min\{k_{i,1}-\pi^2, k_{i,2} - \frac{1}{2M_i}(1+\sum_{j\in\mathcal{N}_i} T_{ij}^2), \\ &\quad k_{i,3} - \frac{1}{2M_i} - \frac{1}{2T_{CH_i}} - \frac{1}{2M_i}\sum_{j\in\mathcal{N}_i}(T_{ij}b_ik_{i,1},k_{i,2})^2, \\ &\quad k_{i,4} - \frac{1}{2T_{CH_i}} - \frac{1}{2M_i}\sum_{j\in\mathcal{N}_i}(T_{ij}b_i'(k_{i,1},k_{i,2},k_{i,3}))^2\}.\end{aligned}$$

If the small-gain condition is satisfied; i.e.,

$$\gamma_i < 1, \quad \forall i \in V, \tag{3}$$

then the closed-loop system is globally exponentially stable. Note that (3) can be always realized by *sequentially* choosing sufficiently large $k_{i,1}, k_{i,2}, k_{i,3}, k_{i,4}$.

III. Stability constrained incentive mechanisms

In this section, we will introduce the problem of stability constrained incentive design. To solve the problem, we will propose a reward based mechanism where the award amount received by each control authority is proportional to one's contribution to the system stability. Finally, we will formally analyze the proposed incentive scheme.

A. Stability constrained incentive design

The system operator aims to ensure the frequency stability. However, the control authorities are self-interested and may seek for different and even partially conflicting subobjectives. So, the closed-loop system may not be stable. Consequently, the system operator aims to incentivize self-interested control authorities such that the gain functions are contraction mappings. This introduces the problem of *stability constrained incentive design*.

B. Stability constrained incentive mechanism

In what follows, we will propose a reward based scheme built on fixed-prize raffles in [13] to address the problem of stability constrained incentive design. The reward based scheme introduces a non-cooperative game among the control authorities and parameterized by the reward. After that, we will identify a lower bound of the reward such that the induced Nash equilibrium enforces the stability condition (3).

1) Low level decision making - Nash equilibrium: The system operator adopts $s_i = \frac{1}{\gamma_i}$ with $s_i \geq 0$ as the contribution made by control authority i to system stability. In order to stimulate contribution, the system operator provides a reward with some fixed amount $R > 0$. We assume that the system operator accepts the deficit financing of an amount δ. The value R together with some constant $\delta \in (0, R)$ are publicized to all the control authorities. The system operator allocates the rewards to the control authorities according to their contributions. In particular, the system operator allocates the portion $\frac{s_i}{\bar{s}}R$ to control authority i where $\bar{s} \triangleq \mathbf{1}^T s$. That is, each control authority can get a larger portion of the reward if he makes a bigger contribution to the system stability. The reward allocation scheme introduces competitiveness among the control authorities. In order to incentivize the control authorities to increase their contributions, the system operator pays for the additional reward $\bar{s} - R$ if $\bar{s} \geq R - \delta$. If the total contribution exceeds the reward value; i.e., $\bar{s} \geq R - \delta$, then control authority receives payoff $R\frac{s_i}{\bar{s}} + h_i(\bar{s} - R)$ from the system operator. Since control authorities are heterogeneous, so h_i may not be identical. If the total contribution $\bar{s}$ does not reach $R - \delta$; i.e., $\bar{s} < R - \delta$, then control authorities receive zero award from the system operator.

As a result, the utility of control authority i is given by the following:

$$U_i(s) = R\frac{s_i}{\bar{s}} + h_i(\bar{s} - R) - s_i,$$

if $\bar{s} \geq R - \delta$; otherwise, $U_i(s) = 0$. In the utility U_i, the function h_i satisfies the following assumption:

Assumption 3.1: The function $h_i : \mathbb{R} \to \mathbb{R}$ with $h_i(0) = 0$ is twice differentiable, non-decreasing and concave. There is $\xi > 0$ such that $\sum_{i\in V} \frac{dh_i(s)}{d\nu} < 1$ for all $s \geq \xi$.

Each control authority aims to maximize its own utility. This induces a non-cooperative game among control authorities. We will use Nash equilibrium as the solution notion of the game.

Definition 3.1: The joint decision s^* is a Nash equilibrium if $U_i(s_i, s^*_{-i}) \leq U_i(s^*)$ for any $s_i \geq 0$.

We denote Nash equilibrium as $s^*(R) \triangleq \{s^*_i(R)\}_{i\in V}$ where the dependency of Nash equilibrium on R is highlighted. We will investigate the existence and uniqueness of Nash equilibrium.

After computing a Nash equilibrium $s^*(R)$, each control authority i commits to $s^*_i(R)$ by a distributed controller u_i such that $\gamma_i = \frac{1}{s^*_i(R)}$.

The reward based incentive mechanism is summarized as follows:

1) The system operator chooses R, δ and h_i;
2) The control authorities collectively determine a Nash equilibrium $s^*(R)$;
3) By following the steps in Section II, each control authority i chooses a distributed controller u_i such that $\gamma_i \leq \frac{1}{s^*_i(R)}$;
4) Each control authority implements distributed controller u_i.

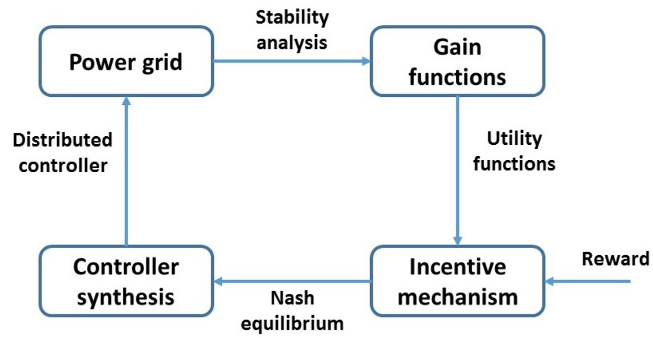

Fig. 1: The framework of stability constrained mechanism design.

2) High level decision making - Social optimum: In Theorem 3.1, we will show that for any $R \geq R_{\min} \triangleq \hat{R}(1)$ with $\hat{R}_L$ in (7), $s^*(R)$ induces distributed controllers which enforces (3). Combining this with maximizing the stability margin, the goals of the system operator are formalized as follows:

$$\max_{R\geq R_{\min}} U_{op}(R) \triangleq \mathbf{1}^T s^*(R) - \sum_{i\in V} h_i(\mathbf{1}^T s^*(R) - R) - R. \tag{4}$$

The optimal solution R^* of (4) is referred to as social optimum.

C. Discussion

There are a couple of reasons for us to study the proposed reward based scheme. First, a similar scheme was experimented in Indian to reduce traffic congestion and see [12]. Secondly, our scheme can be viewed as a non-cooperative game parameterized by the reward R. The parametric structure allows us to derive a simple lower bound for R in Theorem 3.1 such that the induced Nash equilibrium can ensure the stability condition (3). This provides a guideline for the system operator to choose the reward value.

Our reward based incentive scheme is built on fixed-prize raffles in [13]. However, the decision variables in our scheme are not upper bounded. This allows us to perform the sensitivity analysis of Nash equilibrium with respect to the reward. This set of analysis is novel and necessary to ensure (3).

We would like to introduce a set of notations for next section. Let G^* be the optimal solution to the following optimization problem:

$$\max_{G\geq 0} \sum_{i=1}^{N} h_i(G) - G. \tag{5}$$

By Assumption 3.1, there is $G' > 0$ such that $\sum_{i=1}^{N} h_i(G) - G < 0$ for all $G \geq G'$. Recall that $\sum_{i=1}^{N} h_i(0) = 0$. This implies that G^* exists. The quantity $R_L \geq \delta$ is sufficiently large such that

$$-1 + \frac{R_L}{R_L + G^*} + \frac{dh_i(G^*)}{d\nu} > 0, \quad \forall i \in V. \tag{6}$$

Given $\Delta > 0$, we let

$$\hat{R}_L(\Delta) \triangleq \max\{R_L, \frac{(1 - \frac{1}{2}\frac{dh_i(G^*)}{d\nu})G^*}{\frac{1}{2}\frac{dh_i(G^*)}{d\nu}}, \delta + \frac{\Delta}{\frac{dh_i(G^*)}{d\nu}} + \sqrt{\frac{\Delta^2}{(\frac{dh_i(G^*)}{d\nu})^2} + \frac{2\delta\Delta}{\frac{dh_i(G^*)}{d\nu}}}\}. \tag{7}$$

Let $\Theta_{\max}(R) \subseteq V$ be the set of $i \in V$ such that $s_i^*(R) \geq s_j^*(R)$ for all $j \neq i$. Let $\Theta_{\min}(R) \subseteq V$ be the set of $i \in V$ such that $s_i^*(R) \leq s_j^*(R)$ for all $j \neq i$.

Let $B_i(R) > 0$ such that $R + h_i(B_i(R) - R) - B_i(R) < 0$.

D. Analysis

1) Existence and uniqueness of Nash equilibrium: The following lemma shows the existence and uniqueness of Nash equilibrium.

Lemma 3.1: Given any $R > 0$, there is a unique Nash equilibrium.

Proof: Assume $s_i^*(R) \geq B_i(R)$ for some i. Then we have

$$U_i(s^*(R)) \leq R + h_i(B_i(R) - R) - B_i(R) < 0.$$

Consider $\bar{s}$ where $\bar{s}_{-i} = s_{-i}^*(R)$ and $\bar{s}_i = 0$. Since $U_i(\bar{s}) \geq 0$, then we reach a contradiction and thus $s_i^*(R) < B_i(R)$. Hence, $s^*(R)$ is identical to the following game: $\max_{s_i} U_i(s)$ s.t. $s_i \in [0, B_i(R)]$. In this game, the utility functions are concave and the decision variables lie in compact sets. Hence, $s^*(R)$ exists. The uniqueness of Nash equilibrium can be proven by following similar arguments of Lemma 3 in [13]. ■

2) Analysis of Nash equilibrium: Theorem 3.1 summarizes a set of properties of Nash equilibrium for the low level. In particular, (P1) means that the total contribution is within a constant distance to the reward. This indicates that the competitiveness created by the incentive mechanism promotes the total contribution. (P2) further examines the positive externality of s_i^* with respect to R. Notice that $L_i(R)$ is strictly increasing in R. (P3) then indicates that $s_i^*(R)$ could be beyond any given $\Delta > 0$ by choosing a reward larger than $\hat{R}_L(\Delta)$. The property has been used to determine the minimum reward $R_{\min}$. (P4) shows that some $s_i^*(R)$ increases at any R. The relation (8) in (P5) indicates that each control authority receives a non-trivial portion of the reward at any Nash equilibrium, demonstrating partial fairness of the incentive scheme.

Theorem 3.1: The following properties hold for Nash equilibrium $s^*(R)$:

- (P1) Given any $R > 0$, $R + G^* \geq \mathbf{1}^T s^*(R) \geq R - \delta$;
- (P2) Given any $R \geq R_L$, $s_i^*(R) > L_i(R) \triangleq \frac{(R-\delta)^2}{R}(-1 + \frac{dh_i(G^*)}{d\nu} + \frac{R}{R+G^*}) > 0$;
- (P3) Given any $\Delta > 0$, $s_i^*(R) \geq \Delta$ for any $R \geq \hat{R}_L(\Delta)$ and $i \in V$;
- (P4) Given any $R \geq R_L$, there is some $i \in V$ such that $\frac{ds_i^*(R)}{dR} > 0$;
- (P5) It holds that

$$\frac{s_i^*(R)}{\mathbf{1}^T s^*(R)} R \geq \frac{L_i(R)}{R + G^*} > 0, \tag{8}$$

$$\liminf_{R \to +\infty} \frac{s_i^*(R)}{\bar{s}^*(R)} \geq \frac{dh_i(G^*)}{d\nu}. \tag{9}$$

Proof: In the proof, we will drop the dependency of s^* on R and use the notation $\bar{s}^* \triangleq \mathbf{1}^T s^*$.

The property (P1) is a direct result of Lemma 4 and the arguments after Corollary 1 in [13].

Given any $R \geq 0$, the Nash equilibrium $s^*(R)$ must satisfy the first-order condition:

$$\frac{dU_i(s^*)}{ds_i} \leq 0. \tag{10}$$

Let us proceed to show (P2). The first-order partial derivative of U_i with respect to s_i at $s^* = s^*(R)$ is given by:

$$\frac{dU_i(s^*)}{ds_i} = -1 + R\frac{\bar{s}^* - s_i^*}{(\bar{s}^*)^2} + \frac{dh_i(\bar{s}^* - R)}{d\nu}. \tag{11}$$

Since $\frac{dh_i}{d\nu}$ is non-increasing, it follows from (P1) and (11) that

$$\begin{aligned}\frac{dU_i(s^*)}{ds_i} &\geq -1 + R\frac{\bar{s}^* - s_i^*}{(\bar{s}^*)^2} + \frac{dh_i(G^*)}{d\nu} \\ &= -1 + R(\frac{1}{\bar{s}^*} - \frac{s_i^*}{(\bar{s}^*)^2}) + \frac{dh_i(G^*)}{d\nu} \\ &\geq -1 + R(\frac{1}{R + G^*} - \frac{s_i^*}{(R-\delta)^2}) + \frac{dh_i(G^*)}{d\nu},\end{aligned} \tag{12}$$

where in the last inequality we use (P1). If $s_i^* \leq L_i(R)$, it follows from (12) that $\frac{dU_i(s^*)}{ds_i} > 0$, contradicting the first-order condition (10). Hence, $s_i^*(R) > L_i(R)$. This completes the proof of (P2).

We now proceed to show (P3). Since $R \geq \hat{R}_L(\Delta)$, we have $R \geq \frac{(1-\frac{1}{2}\frac{dh_i(G^*)}{d\nu})G^*}{\frac{1}{2}\frac{dh_i(G^*)}{d\nu}}$ and then

$$-1 + \frac{dh_i(G^*)}{d\nu} + \frac{R}{R + G^*} \geq \frac{1}{2}\frac{dh_i(G^*)}{d\nu}. \tag{13}$$

Combining (P2) and (13) renders

$$s_i^*(R) \geq \frac{(R-\delta)^2}{R}\frac{1}{2}\frac{dh_i(G^*)}{d\nu}. \tag{14}$$

Since $R \geq \hat{R}_L \geq \delta + \frac{\Delta}{\frac{dh_i(G^*)}{d\nu}} + \sqrt{\frac{\Delta^2}{(\frac{dh_i(G^*)}{d\nu})^2} + \frac{2\delta\Delta}{\frac{dh_i(G^*)}{d\nu}}}$, the right-hand side of (14) is greater than or equal to Δ and so is $s_i^*(R) \geq \Delta$. This completes the proof of (P3).

By (P2) and (10), the first-order partial derivative of U_i with respect to s_i vanishes at $s^*(R)$. That is,

$$\frac{dU_i(s^*)}{ds_i} = -1 + R\frac{\bar{s}^* - s_i^*}{(\bar{s}^*)^2} + \frac{dh_i(\bar{s}^* - R)}{d\nu} = 0. \tag{15}$$

We now proceed to derive an expression for $\frac{ds_i^*}{dR}$ from (15). Notice that

$$\begin{aligned}\frac{d^2U_i(s^*)}{ds_i^2} &= -2R\frac{\bar{s}^*-s_i^*}{(\bar{s}^*)^3}+\frac{d^2h_i(\bar{s}-R)}{d^2\nu},\\ \frac{d^2U_i(s^*)}{ds_ids_j} &= -2R\frac{2s_i^*-\bar{s}^*}{(\bar{s}^*)^3}+\frac{d^2h_i(\bar{s}^*-R)}{d^2\nu},\\ \frac{d^2U_i(s^*)}{ds_idR} &= \frac{\bar{s}^*-s_i^*}{(\bar{s}^*)^2}-\frac{d^2h_i(\bar{s}^*-R)}{d^2\nu}.\end{aligned} \tag{16}$$

It follows from the implicit function theorem and (16) that

$$\begin{aligned}&-\begin{bmatrix}\frac{d^2U_1(s^*(R))}{d^2s_1} & \cdots & \frac{d^2U_i(s^*(R))}{ds_1ds_N}\\ \vdots & \ddots & \vdots\\ \frac{d^2U_N(s^*(R))}{ds_1ds_N} & \cdots & \frac{d^2U_N(s^*(R))}{d^2s_N}\end{bmatrix}\begin{bmatrix}\frac{ds_1^*(R)}{dR}\\ \vdots\\ \frac{ds_N^*(R)}{dR}\end{bmatrix}\\ &=\begin{bmatrix}\frac{d^2U_1(s^*(R))}{ds_1dR}\\ \vdots\\ \frac{d^2U_N(s^*(R))}{ds_NdR}\end{bmatrix}>0.\end{aligned} \tag{17}$$

For any $s \geq 0$ and $R \geq 0$, we have $\frac{d^2U_i(s)}{d^2s_i} < 0$ and $\frac{d^2U_i(s)}{ds_idR} > 0$. Pick any $i' \in \Theta_{\min}(R)$. Then $2s_{i'}^*(R) - \bar{s}^*(R) < 0$ and thus $\frac{d^2U_{i'}(s^*)}{ds_{i'}ds_j} < 0$. Hence, we have

$$-\sum_{j\in V}\frac{d^2U_{i'}(s^*)}{ds_{i'}ds_j}\frac{ds_j^*(R)}{dR}=\frac{dU_{i'}(s^*)}{ds_{i'}dR}>0. \tag{18}$$

The relation (18) implies that there is at least one $i \in V$ such that $\frac{ds_i^*(R)}{dR} > 0$. This completes the proof of (P4).

By using (P1) and (P2), we have (8). Take the limit on R at both sides of (8), we reach (9). ■

3) Analysis of social optimum: The following theorem examines the properties of U_{op}. (P6) verifies the uniform boundedness of U_{op}.

Theorem 3.2: The following properties hold for U_{op}:

- (P6) $\max\{-\delta - \sum_{i\in V} h_i(G^*), \frac{\frac{dh_i(G^*)}{d\nu}-1}{N-\frac{dh_i(G^*)}{d\nu}}R - \sum_{i\in V}\frac{dh_i(-\delta)}{d\nu}\} \leq U_{op}(R) \leq \min\{G^* - \sum_{i\in V} h_i(-\delta), \frac{\frac{dh_i(0)}{d\nu}-1}{N-\frac{dh_i(0)}{d\nu}}R - \sum_{i\in V}\frac{dh_i(G^*)}{d\nu}\}$.

Proof: Sum (11) over i and we have

$$-N+\frac{R(N-1)}{\bar{s}^*}+\sum_{i\in V}\frac{dh_i(\bar{s}^*-R)}{d\nu}=0.$$

This implies the following relation:

$$\bar{s}^*=\frac{R(N-1)}{N-\sum_{i\in V}\frac{dh_i(\bar{s}^*-R)}{d\nu}}.$$

Since $\frac{dh_i(\bar{s}^*-R)}{d\nu}$ is non-increasing, then we have

$$\begin{aligned}&\frac{\frac{dh_i(G^*)}{d\nu}-1}{N-\frac{dh_i(G^*)}{d\nu}}R-\sum_{i\in V}\frac{dh_i(-\delta)}{d\nu}\leq U_{op}(R)\\ &\leq\frac{\frac{dh_i(0)}{d\nu}-1}{N-\frac{dh_i(0)}{d\nu}}R-\sum_{i\in V}\frac{dh_i(G^*)}{d\nu}.\end{aligned} \tag{19}$$

Then (19) and (P1) imply the desired result (P6). ■

IV. Conclusions

In this paper, we have formulated the problem of stability constrained incentive design. We have proposed a reward based incentive scheme and determined the lower bound of the reward such that the induced Nash equilibrium can ensure grid stability. The proposed incentive scheme requires control authorities to disclose limited private information.

References

[1] L. Chen, N. Li, L. Jiang, and S. Low. *Optimal Demand Response: Problem Formulation and Deterministic Case*, pages 63–85. Control and Optimization Methods for Electric Smart Grids. Springer, 2011.

[2] C. Dwork. Differential privacy. In *3rd International Colloquium on Automata, Languages and Programming*, pages 1–12, 2006.

[3] C. Farkas and S. Jajodia. The inference problem: A survey. *ACM SIGKDD Explorations Newsletter*, 4(2):6–11, 2002.

[4] C.W. Gellings. *The Smart Grid: Enabling Energy Efficiency and Demand Response*. U.K.: CRC Press, 2009.

[5] C.W. Gellings and J.H. Chamberlin. *Demand side management: Concepts and methods*. The Fairmont Press, 1988.

[6] D.J. Glover, M.S. Sarma, and T.J. Overbye. *Power system analysis and design*. Cengage Learning, 2012.

[7] R. Johari and J.N. Tsitsiklis. Efficiency loss in a network resource allocation game. *Mathematics of Operations Research*, 29(3):407–435, 2004.

[8] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. John Wiley and Sons, 1995.

[9] I. Kumar and D. Kothari. Recent philosophies of automatic generation control strategies in power systems. *IEEE Transactions on Power Systems*, 20(1):346 – 357, 2005.

[10] P. Kundur, N.J. Balu, and M.G. Lauby. *Power system stability and control*. McGraw-Hill, 1994.

[11] S. McLaughlin, P. McDaniel, and W. Aiello. Protecting consumer privacy from electric load monitoring. In *18th ACM Conference on Computer and Communications Security*, pages 87–98, 2011.

[12] D. Merugu, B. Prabhakar, and N. Rama. An incentive mechanism for decongesting the roads: A pilot program in bangalore. In *ACM NetEcon Workshop*, 2009.

[13] J. Morgan. Financing public goods by means of lotteries. *Review of Economic Studies*, 67:761–784, 2000.

[14] N. Nisan and A. Ronen. Algorithmic mechanism design. In *31st Annual ACM Symposium on Theory of computing*, pages 129–140, 1999.

[15] J. Le Ny and G. Pappas. Differentially private filtering. *IEEE Transactions on Automatic Control*, 59(2):341–354, 2014.

[16] D.C. Parkes and J. Shneidman. Distributed implementations of Vickrey-Clarke-Groves mechanisms. In *International Joint Conference on Autonomous Agents and Multi-agent Systems*, pages 261–268, 2004.

[17] A. Petcu, B. Faltings, and D.C. Parkes. MDPOP: Faithful distributed implementation of efficient social choice problems. In *International Joint Conference on Autonomous Agents and Multi-agent Systems*, pages 1397–1404, 2006.

[18] A. Roth. New algorithms for preserving differential privacy. Technical report, Ph.D. dissertation, Carnegie Mellon University, 2010.

[19] L. Sankar, S.R. Rajagopalan, S. Mohajer, and H.V. Poor. Smart meter privacy: A theoretical framework. *IEEE Transactions on Smart Grid*, 4(2):837–846, 2013.

[20] T. Tanaka, F. Farokhi, and C. Langbort. A faithful distributed implementation of dual decomposition and average consensus algorithms. In *IEEE Conference on Decision and Control*, pages 2985–2990, 2013.

[21] P. Vytelingum, T. Voice, S. Ramchurn, A. Rogers, and N. Jennings. Agent-based microstorage management for the smart grid. In *International Conference on Autonomous Agents and Multiagent Systems*, pages 39– 46, 2010.

[22] A.J. Wood and B.F Wollenberg. *Power Generation, Operation and Control*. J. Wiley and Sons, New York,, 1996.

[23] F.F. Wu, K. Moslehi, and A. Bose. Power system control centers: Past, present and future. *Proceedings of the IEEE*, 93(11):1890 – 1908, 2005.

[24] M. Zhu. Distributed demand response algorithms against semi-honest adversaries. In *IEEE PES General Meeting*, 2014. No. 943.