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A Market Mechanism for Electric Distribution Networks

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Abstract— **To encourage end-users to participate in the transformation of the power grid into a more distributed, adaptive and resilient one, an efficient electricity market, especially in distribution networks, plays an important role. However, the externalities associated with the power flow and network operating constraints constitute a significant barrier to form such markets.[1] Traditionally, there is a central regulator to determine locational marginal prices in order to compensate the externalities. In this paper, we present a decentralized market mechanism for a radial distribution network which internalize the externalities within private decisions. The market mechanism defines trading rules that efficiently allocate the externalities to individual bilateral transactions in the network. Specifically, we focus on the external costs associated with voltage constraints and line losses. We show that a competitive market could be established for distribution services and electricity to achieve a social optimum within a power pool.**

I. Introduction

To transfer the power grid into a distributed, adaptive, and resilient grid requires great participation from end-users, such as demand response. To this end, it involves sophisticated design of the energy management systems, incentives, and/or a viable trading market in order to guarantee the grid's efficiency and reliability. Among those, pricing policies especially in distribution networks play a fundamental role in shaping consumers behavior. According to Kirchhoff's law, any local change (e.g. power injection) in the power network will affect the global power network state (e.g. either power flow or voltage or both). As a result, the local decisions of users usually cause external costs associated with power flows, voltages, line losses, etc. These external costs constitute a significant barrier to form efficient markets for electricity networks.

There have been many research efforts in developing pricing schemes either for electricity wholesale markets or retail markets. The markets are either at day-ahead or in real-time, aiming to either smooth the demand profile or call on users' action to compensate the fluctuations of renewable energy [1]-[9] . However, most of the work either omits the power network structure or uses the DC approximation model. Thus the pricing policies could not reflect the externalities associated with the AC power networks. Some literature applies AC optimal power model and proposes locational marginal price [5], [6], [9]. Though the locational marginal price is an efficient pricing scheme to promote social welfare, the issues are that: i) it requires a central regulator to determine the efficient locational marginal price to compensates the externalities, no matter whether the central regulator uses a centralized or distributed method to determine such prices; ii) there is a lack of deep understanding of how the externalities are reflected in the locational marginal prices, or in other words, how the externalities are allocated to individuals in the network through the locational prices is unclear. As a result, it remains an open question how to design an decentralized market to allow individuals to trade electricity/power with each other without affecting the global network performance.

In this paper we present a market mechanism for a radial distribution network which internalizes the external costs within private decisions, such as local trading transactions. We define trading rules which efficiently allocate the external costs to individual bilateral transactions in the network. [10] presents a market design for transmission systems where externalities of transmission line congestion and transmission losses play a central role. However, in distribution networks which have the most of end-users, voltage operation constraints (which can be interpreted as another type of congestion) play a more significant role than transmission line congestion because it affects the power delivery and power quality.[2] Therefore in this paper we focus on the external cost associated with voltage constraints and distribution line losses. The main idea of this paper is similar to the work of [10] where a market for transmission network is proposed. Specifically, we adopt the notion of voltage rights and introduce a trading rule that specifies the voltage rights and the line-loss compensation for each transaction of power dispatch. The trading rule takes into account the effects of individual transactions on the voltages and line losses across the entire distribution network. With these property rights and the trading rules, we demonstrate that a competitive equilibrium can achieve a social optimum. The proposed mechanism serves as an initial step to provide guidelines for creating an efficient and decentralized market for electric distribution services.

The rest of the paper is organized as follows: Section II presents an AC power flow model for distribution networks; Section III presents a simple example with only 3 nodes to illustrate the main idea of this paper; Section IV provides a market mechanism and trading rules by neglecting the power losses; Section V extends the idea to the full AC network model by considering the power losses; and Section VII

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[1]In economics, an externality is the cost or benefit that affects a party who did not choose to incur that cost or benefit.

[2]In distribution networks, it is required to maintain acceptable voltages (plus or minus 5% around nominal values) at all nodes along the distribution feeder.

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discusses the limitation of the results and future work and concludes the paper.

II. Network Model

In this section, we advocate the use of branch flow model for a radial distribution network. Branch flow models focus on currents and power flows on individual branches [11]. They have been used mainly for modeling distribution circuits which tend to be radial.

A. Branch flow model for radial distribution network

Consider a radial distribution circuit that consists of a set N of nodes(/buses) and a set E of distribution lines connecting these nodes. We index the nodes in N by $i = 0, 1, \ldots, n$, where node 0 represents the substation (or the feeder) and other nodes in N represent branch nodes. We also denote a line in E by the pair (i, j) of nodes it connects where j is closer to the feeder 0. We call j the parent of i, denoted by $\pi(i)$, and call i the child of j. Denote the child set of j as $\delta(j) := \{i : (i, j) \in E\}$. Thus a link (i, j) can be denoted as $(i, \pi(i))$. We also denote the unique path from node 0 to node i as $\mathcal{P}_i$, i.e. $\mathcal{P}_i := \{k : k \text{ is on the path from node 0 to node } i\}$, and the descent set of node i as Δ_i, i.e., $\Delta_i := \{k : i \in \mathcal{P}_k\}$. We let $i \in \mathcal{P}_i$, $0 \notin \mathcal{P}_i$, $i \in \Delta_i$.

For each line $(i, \pi(i)) \in E$, let I_i be the complex current flowing from nodes i to $\pi(i)$, $z_i = r_i + \mathbf{i}x_i$ the impedance on the link, and $S_i = P_i + \mathbf{i}Q_i$ the complex power flowing from node i to node $\pi(i)$. On each node $i \in N$, let V_i be the complex voltage, $s_i^d = p_i^d + \mathbf{i}q_i^d$, $s_i^g = p_i^g + \mathbf{i}q_i^g$ be the complex power consumption and power generation, and $s_i := p_i + \mathbf{i}q_i := s_i^g - s_i^d$ be the net power injection. As customary, we assume that the complex voltage V_0 on the substation node is given and fixed. Define $\ell := |I_{ij}|^2$, $v_i := |V_i|^2$.

The branch flow model, first proposed in [11] models power flows in a steady state in a radial distribution network:

$$S_i = s_i^g - s_i^d + \sum_{k \in \delta(i)} \left(S_k - (r_k + \mathbf{i}x_k)\ell_i\right), i = 0, \ldots, n, \quad (1a)$$

$$v_i = v_{\pi(i)} + 2(r_i P_i + x_i Q_i) - (r_i^2 + x_i^2)\ell_i, \quad (1b)$$
$$i = 1, \ldots, n,$$

$$\ell_i = \frac{P_i^2 + Q_i^2}{v_i}, i = 1, \ldots, n \quad (1c)$$

where $S_0 := 0 + \mathbf{i}0$. Notice that $s_0^g - s_0^d$ can be interpreted as the total power injection into the distribution network from the main grid through the feeder node 0. Equations in (1) define a system of equations in the variables $(P, Q, \ell, v) := (P_{ij}, Q_{ij}, \ell_{ij}, (i, j) \in E, i = 1, \ldots, n)$, which do not include phase angles of voltages and currents. Given an (P, Q, ℓ, v) these phase angles can be uniquely determined for radial networks [12].

In addition to power flow equations (1), voltage magnitudes must be maintained within certain operating constraints:

$$\underline{v}_i \le v_i \le \bar{v}_i, i = 1, \ldots, n. \quad (2)$$

B. Social welfare maximization–market objective

The market objective considered in this paper is to maximize the social welfare–minimize the power generation costs $C_i(p_i^g)$, the power losses $r_{ij}\ell_{ij}$, and maximize the user utilities $B_i(p_i^d)$:[3]

$$\begin{aligned} \min \quad & \sum_{i=0}^{n} C_i(p_i^g) - \sum_{i=0}^{n} B_i(p_i^d) + \sum_{(i,j) \in E} r_{ij}\ell_{ij} \quad (3) \\ \text{s.t.} \quad & (1), (2) \\ \text{over} \quad & p_i^g \in [\underline{p}_i^g, \bar{p}_i^g], p_i^d \in [\underline{p}_i^d, \bar{p}_i^d], i = 0, \ldots, n \\ & P_i, Q_i, \ell_i, v_i, i = 1, \ldots, n \end{aligned}$$

C. Linear approximation of the network model

Real distribution circuits usually have very small r, x, i.e. $r, x << 1$, while $v \sim 1$. Thus real and reactive power losses are typically much smaller than power flows P_{ij}, Q_{ij}. Following [13], we neglect the higher order real and reactive power loss terms in (1) by setting $\ell_{ij} = 0$ and approximate P, Q, v using the following linear approximation, known as Simplified Distflow introduced in [13].

$$S_i = s_i^g - s_i^d + \sum_{k \in \delta(i)} S_k, i = 0, \ldots, n, \quad (4a)$$

$$v_j = v_i - 2(r_i P_i + x_i Q_i), (i, j) \in E \quad (4b)$$

From (4), we can derive that the voltages $\{v_k\}_{k=1,\ldots,N}$ and net power injections $\{s\}_{i=1,\ldots,N}$ satisfy the following equation:[4]

$$v_k = \sum_{i=1}^{n} R_{ki}(p_i^g - p_i^d) + \sum_{i=1}^{n} X_{ki}(q_i^g - q_i^d) + v_0,$$

where $R_{ki} := 2\sum_{h \in \mathcal{P}_k \cap \mathcal{P}_i} r_h$, and $X_{ki} := 2\sum_{h \in \mathcal{P}_k \cap \mathcal{P}_i} x_h$. As we assume that (q_i^g, q_i^d) are fixed for $i \in N$, we define $v_k^{nom} := \sum_{i=1}^{n} X_{ki}(q_i^g - q_i^d) + v_0$. Therefore, for $k = 1, \ldots, n$,

$$v_k = \sum_{i=1}^{n} R_{ki}(p_i^g - p_i^d) + v_k^{nom}. \quad (5)$$

In this paper, we will use the linear approximation model to illustrate how voltage constraints shape the market equilibrium and how we can design a market to let the externalities associated with voltage constraints internalized within private transactions.

III. An illustrative example

In this section, we present a simple case to highlight some important characteristics of the externalities associated with the voltage constraints. For illustrative purposes, we consider a simple distribution line with three nodes as shown in Figure 1 and we apply the linear approximation model as shown in Section II-C. We assume that there is a generator at

[3] In the optimization problem, we treat reactive power (q_i^g, q_i^d) as given constants rather than decision variables. The reason is for expository simplicity. The mechanisms developed in this paper can be extended to the case where reactive power consumption and generation are decision variables.

[4] The detailed derivation is given in [14].

node 0, a consumer at node 1, and a consumer at node 2. We further assume that consumers at node 1 and 2 have demand capacities of $\bar{p}_1^d = 9$ and $\bar{p}_2^d = 9$ and constant marginal utilities of $b_1 = 18$ and $b_2 = 20$ respectively. The generator at node 0 has an inverse supply function $\lambda(p^g) = 2p^g + 4$, where p^g is the electricity supply. We further assume that the two distribution lines, labeled $(0, 1)$ and $(1, 2)$, have identical electrical characteristics (i.e. line impedance) and that line losses are negligible. We consider the market where the generator and consumers can trade (selling/buying) electricity with each other bilaterally. For example, generator at node 0 and consumer at node 1 (or at node 2) can determine the amount of power traded between them make their own private decision regarding selling/buying electricity. Let $p_{0\to1}$ and $p_{0\to2}$ denote the power that generator at node 0 sells to node 1 and node 2 respectively.

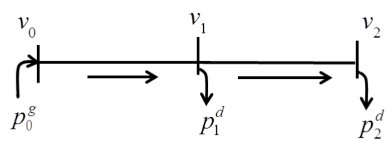

Fig. 1. A simple distribution network with 3 nodes.

Firstly, suppose that the voltage operation constraints are ignored. The social optimal trading transaction would be $p_{0\to1} = 0$ and $p_{0\to2} = 8$, i.e. generator at node 0 sell 8 units of power to the consumer at node 2 and does not sell any power to the consumer at node 1. The marginal cost of electricity at node 0 is 20 per unit. However, if there is a voltage constraint on node 2, e.g. $v \geq \underline{v}$, this brings an additional constraint on the power consumption on node 1 and 2, which can be written in the form of $p_1^d + 2p_2^d \leq \bar{p}$ according to equation (5). Suppose $\bar{p} = 10$. The previous transaction would violate the voltage constraint. As a result, it is not be feasible. Suppose as a result of market trading, the transaction is $p_{0\to1} = 4$ and $p_{0\to2} = 3$. Under this transaction, the voltage constraint is satisfied and the marginal price of electricity at node 0 is 18. At this price, the net revenue at node 1 is zero. Although it would be profitable to increase the consumption at node 2, its consumption is constrained by the voltage constraint. As a result, the trading decisions would be stuck at $p_{0\to1} = 4$ and $p_{0\to2} = 3$. However, with the additional constraint, the socially optimal transaction would be $p_{0\to1} = 2$ and $p_{0\to2} = 4$.

The above example illustrates that physical constraints (voltage constraints) causes the private cost and the social cost diverge from each other in electricity trading transactions. The bilateral transaction between node 0 and node 1 affects the bilateral transaction between node 0 and node 2, and vice versa. In economics, this effect represents externalities. Traditionally, externalities in electric networks are handled by a central system operator. However, an alternative approach is to design new property rights and create markets for these rights so that the external effects associated with a transaction can be internalized in private decisions. For instance, we can introduce voltage rights and establish the associated trading rules to define the quantity of voltage rights that a trader needs to acquire in order to transfer power from one node to another node. As a result, each bilateral transaction is aware of the global network constraints and is able to reach the socially-optimal decisions. For example, in our previous 3-node system, if we introduce voltage rights on node 2 and require each transactions to purchase voltage rights for selling/buying power, then the transactions $p_{0\to1}$ and $p_{0\to2}$ would deviate from $(4, 3)$ and possibly reach the socially optimal ones. However, it remains as a question how to define such voltage rights and trading rules. In the rest of the paper, we will adopt this idea to design a decentralized electricity market for distribution networks to compensate the externalities associated with voltage constraints as well as power losses.

IV. A Market Mechanism for voltage constraints

For expository simplicity, we begin by assuming there are no distribution line losses and using the linear approximation model in (4). This allows us to focus on the voltage constraints in this section. In the next section, we will extend the idea to the nonlinear AC power flow model where distribution line losses are considered. With the linear approximation model (4), the social welfare maximization problem is given as:

$$\begin{aligned} \max \quad & \sum_{i\in N} B_i(p_i^d) - \sum_{i\in N} C_i(p_i^g) \qquad (6) \\ \text{s.t.} \quad & (2), (4) \\ \text{over} \quad & p_i^g \in [\underline{p}_i^g, \bar{p}_i^g], p_i^d \in [\underline{p}_i^d, \bar{p}_i^d], i = 0, \ldots, n \\ & P_i, Q_i, \ell_i, v_i, i = 1, \ldots, n \end{aligned}$$

A. A market mechanism

In the following we will propose a decentralized market mechanism for the voltage constraints so that the external effects on the voltage caused by a transaction can be internalized in private decisions. In other words, rather than letting a central regulator such as ISO to solve the social welfare maximization and determine the locational marginal prices, we create a market which explicitly takes into account the voltage constraints so that any bilateral transaction between a pair of nodes will take account of their external effects on the voltages across the entire network.

The mechanism involves the definition of tradable voltage rights and a trading rule that governs the exchange of these rights.

Definition 1 *A voltage right on one node k entitles its owner to the right to send power through adjusting node k's voltage. A fixed set of voltage rights, $\bar{\nu} := \{\bar{\nu}_k, k \in N\backslash\{0\}\}$, $\underline{\nu} := \{\underline{\nu}_k, i \in N\backslash\{0\}\}$, are issued for each node k and these rights are tradable.*[5]

The trading rule specifies the voltage rights that traders must acquire in order to complete an electricity transaction. Electricity transactions usually specify the power transferred

[5]Notice that the voltage at the feeder 0 is fixed as customary.

from one node to another without specifying the actually voltage changes over the network. The following trading rule include the relationship between power transfers and voltage changes so that the voltage rights can be enforced. Without loss of generality, we treat the node 0 as a base point and only need to define the trading rule that governs the transactions between node 0 and every other node in the network.

Definition 2 *The trading rule consists of a set of coefficients $\beta = \{\bar{\beta}_i^k, \underline{\beta}_i^k | i,k \in N\backslash\{0\}\}$, where the value $\bar{\beta}_i^k$, $\underline{\beta}_i^k$ represents the quantity of voltage rights $\bar{\nu}_k$, $\underline{\nu}_k$, respectively on node k that a trader needs to acquire in order to transfer a unit of power from node i to node 0.*

Using this trading rule, we can define trading terms for bilateral transactions between an arbitrary pair of nodes. For instance, to transfer one unit of power from node i to node j requires $\bar{\beta}_{ij}^k \triangleq \bar{\beta}_i^k - \bar{\beta}_j^k$ units of voltage rights $\bar{v}_k$ on node k.

The market mechanism can thus be summarized as $(\bar{v}, \underline{v}, \beta)$. An efficient design of these terms is as follows: $\bar{\nu}_k$ is the voltage upper limit minus the nominal voltage at node k and $\underline{\nu}_k$ is the nominal voltage minus the voltage lower limit at node k, i.e.,

$$\bar{\nu}_k = \bar{v}_k - v_k^{nom}, \underline{\nu}_k = v_k^{nom} - \underline{v}_k; \tag{7}$$

and $\bar{\beta}_i^k$ ($\underline{\beta}_i^k$) is the (negative) voltage-loading factor – (negative) change on the voltage at node k due to a unit power injection at node i. From equation (5), we know that an efficient design of β is

$$\bar{\beta}_i^k = R_{ki}, \underline{\beta}_i^k = -R_{ki}. \tag{8}$$

The following section will further explain why this market design is efficient.

B. Competitive Equilibrium and social optimum

Let us denote by $\bar{\xi}_i$ and $\underline{\xi}_i$ the prices of the voltage rights $\bar{v}_i$ and $\underline{v}_i$ respectively, and by λ_i the price of electricity at node $i \in N$. Denote by λ, p, and ξ the vectors of (λ_i), (p_i^g, p_i^d), and $(\bar{\xi}_i, \underline{\xi}_i,)$ for $i \in N$.

Definition 3 *A competitive equilibrium is a vector (p, λ, ξ) that satisfies the following three conditions:*

i) Each consumer maximize their net benefit and each generator maximize their net revenue given the price λ_i,

$$p_i^d \in arg \max_{p_i^d \in [\underline{p}_i^d, \bar{p}_i^d]} B_i(p_i^d) - \lambda_i p_i^d, \forall i \in N \tag{9}$$

$$p_i^g \in arg \max_{p_i^g \in [\underline{p}_i^g, \bar{p}_i^g]} \lambda_i p_i^g - C_i(p_i^g), \forall i \in N \tag{10}$$

ii) There should be no positive profit to be made by transfer of power from one node to another, i.e.,

$$\lambda_j = \lambda_i + \sum_{k=1}^n \bar{\xi}_k \beta_{ij}^k + \sum_k \underline{\xi}_k \underline{\beta}_{ij}^k, \forall i,j \in N. \tag{11}$$

iii) The price for voltage rights is zero when there is excess supply:

$$\bar{\xi}_i[v_i - \bar{v}_i] = \underline{\xi}_i[v_i - \underline{v}_i] = 0, \forall i \in N. \tag{12}$$

Theorem 1 *Under the market mechanism $(\mathbf{v}, \mathbf{v}, \beta)$, a competitive equilibrium (p, λ, ξ) exists and is socially optimal.*

Proof: Firstly, by using equation (5), the social welfare problem is equivalent to the following optimization problem:

$$\begin{array}{ll} \max & \sum_{i \in N} B_i(p_i^d) - C_i(p_i^g) \\ \text{s.t.} & \sum_{i \in N} p_i^g - \sum_{i \in N} p_i^d = 0 \\ & (2,5) \\ \text{over} & p_i^g \in [\underline{p}_i^g, \bar{p}_i^g], p_i^d \in [\underline{p}_i^d, \bar{p}_i^d], v_i \end{array}$$

For any p where $\underline{p}_i^g \le p_i^g \le \bar{p}_i^g$ and $\underline{p}_i^d \le p_i^d \le \bar{p}_i^d$, the Lagrangian function of this optimization problem is:

$$\begin{aligned} L \;=\; & -\sum_{i=0}^n B_i(p_i^d) + \sum_{i=0}^n C_i(p_i^g) \\ & + \sum_{i=1}^n \omega_i \left(v_i - 2\sum_{j=1}^n R_{ij}(p_j^g - p_j^d) - v_i^{nom} \right) \\ & + \lambda_0 \left(-p_0^g + p_0^d - \sum_{j=1}^n (p_j^g - p_j^d) \right) \\ & + \sum_{i=1}^n \bar{\xi}_i(v_i - \bar{v}_i) + \underline{\xi}_i(-v_i + \underline{v}_i) \end{aligned}$$

Thus $(p, v, \omega, \lambda_0, \xi)$ is a primal-dual optimum if and only if, (p, v) is feasible, and

$$p_i^d \in \arg \max_{p_i^d \in [\underline{p}_i^d, \bar{p}_i^d]} B_i(p_i^d) - \lambda_i p_i^d, \forall i \in N, \tag{13}$$

$$p_i^g \in \arg \max_{p_i^g \in [\underline{p}_i^g, \bar{p}_i^g]} \lambda_i p_i^g - C_i(p_i^g), \forall i \in N, \tag{14}$$

where

$$\lambda_i = \sum_{k=1}^n \omega_k R_{ki} + \lambda_0, \tag{15}$$

$$\omega_i = -\bar{\xi}_i + \underline{\xi}_i, \tag{16}$$

$$\bar{\xi}_i[v_i - \bar{v}_i] = \underline{\xi}_i[\underline{v}_i - v_i] = 0,. \tag{17}$$

Therefore we know that by letting $\bar{\beta}_i^k = R_{ki}$, $\underline{\beta}_i^k = -R_{ki}$, (p, λ, ξ) is a competitive equilibrium.

On the other hand, given a competitive equilibrium (p, λ, ξ), it is easy to show that (13–17) hold. So to show that the competitive equilibrium is optimal, we only need to show that

$$\underline{v}_i \le v_i \le \bar{v}_i, \forall i = 1, \ldots, n. \tag{18}$$

Notice that the quantity of tradable voltage rights demanded can not exceed the total quantity issued in (7). We have

$$\bar{v}_k - v_k^{nom} \ge \sum_i \bar{\beta}_i^k p_i = \sum_i R_{ki} p_i = v_k - v_k^{nom},$$

$$v_k^{nom} - \underline{v}_k \ge \sum_i \underline{\beta}_i^k p_i = -\sum_i R_{ki} p_i = v_k^{nom} - v_k.$$

Thus (18) holds and we show that the competitive equilibrium is social optimal. ■

Applying the preceding theorem, we have an interesting observation of the locational prices λ_i when all the branch nodes $i = 1, \ldots, n$ are load nodes. In this case, equation (5) tells that $v_i \leq v_{\pi(i)} \leq \ldots \leq v_0$. Because in practice $\bar{v}_i > v_0$, we have $v_i < \bar{v}_i$. Therefore $\bar{\xi}_i = 0$ for all $i = 1, \ldots, n$. As a result, we have $\lambda_i > \lambda_{\pi(i)}$ for all $i = 1, \ldots, n$, meaning that the locational price increases along each path from the feeder 0 to the leaves.

V. The case with line losses

For the case with no losses, the distribution network exhibits the linearity property whereby the voltages are linear functions of power injections. The linearity makes it clear to determine the trading rules for voltage rights. In the presence of distribution line losses, things become more complicated. Line losses raise the problem of nonlinearity and even nonconvexity, adding significant complexities to the externality allocation problem. There have been a series of work (see [15] and references therein) studying the convex relaxation of the social welfare maximization problem (3) where the quadratic equality constraint (1) is relaxed to the inequality constraint, $\frac{P_{ij}^2+Q_{ij}^2}{v_i} \leq \ell_{ij}$. Those studies establishes different sufficient conditions to guarantee the exact relaxation and it has been shown that most real radial distribution networks satisfy those sufficient conditions. Thus here we assume that the exact relaxation holds and focus on the convex relaxed problem which justify the use of the Lagrangian duality to develop price policies.

A. A market mechanism

To achieve economic efficiency when there are distribution line losses, it is desirable for the traders to pay for the marginal line losses. Thus a new type of economic rent is created for the line losses. The below market mechanism modifies and extends the previous trading rule by specifying the voltage rights and the compensation for the power losses that are required to acquire for each electricity transaction. Since the node voltages and power losses depend on the actual power flow pattern, the trading rule is state-dependent and varies continuously with time as the system condition evolves.

Formally, the trading rule consists two parts.

- The first part corresponds to the voltage rights which is similar to those in the lossless case. It can be denoted as $\beta := \{\bar{\beta}_i^k, \underline{\beta}_i^k\}$ where $\bar{\beta}_i^k$ ($\underline{\beta}_i^k$, resp.) represents the quantity of voltage rights $\bar{\nu}_k$ ($\underline{\nu}_k$, resp.) on node k that a trader needs to acquire in order to inject a unit of power at node i and deliver it to node 0. For an efficient mechanism, we can define $\bar{\nu}_k$ and $\underline{\nu}_k$ according to (7), and $\bar{\beta}_i^k$ and $\underline{\beta}_i^k$ as:

$$\bar{\beta}_i^k = \frac{(v_k - v_k^{nom})\, R_{ki}}{\sum_{j=1}^n R_{kj} p_j}; \quad \underline{\beta}_i^k = -\frac{(v_k - v_k^{nom})\, R_{ki}}{\sum_{j=1}^n R_{kj} p_j}. \quad (19)$$

  By defining the trading rule in this way, we have

$$\sum_{i=1}^n \bar{\beta}_i^k p_i = v_k - v_k^{nom}, \sum_{i=1}^n \underline{\beta}_i^k p_i = v_k^{nom} - v_k, \quad (20)$$

  Roughly speaking, the coefficient β_i^k can be interpreted as the average loading factor of node i for the voltage on node k.
- The second part corresponds to the economic rent for the distribution line losses. We denote the trading rule as $\phi := \{\phi_i^k : i, k = 1, \ldots, n\}$ where ϕ_i^k represents the quantity of line losses on link $(k, \pi(k))$ that a trader needs to pay rents in order to inject an unit of power at node i and deliver it to node 0. We define the trading rule as:

$$\phi_i^k = \frac{L_k L_{k,i}}{\sum_{j=1}^n L_{k,j} p_j} \quad (21)$$

  Here L_k denote the line losses on $(k, \pi(k))$ and $L_{k,i}$ is given as follows:

$$L_{k,i} \;=\; r_k \begin{cases} \frac{2P_k}{v_k} - \frac{P_k^2+Q_k^2}{v_k^2} R_{ki}, & \text{if } k \in \mathcal{P}_i, \\ -\frac{P_k^2+Q_k^2}{v_k^2} R_{ki}, & \text{otherwise.} \end{cases} \quad (22)$$

  By defining the trading rule in this way, we have

$$\sum_{i=1}^n \phi_i^k p_i = L_k. \quad (23)$$

  Roughly speaking, ϕ_i^k can be interpreted as the average line losses by injecting power at node i.

Remark [Physical interpretation of $R_{k,i}, L_{k,i}$]: If we use the linear approximation model (4) to approximate the power flow and voltage, we have seen in section IV that $R_{k,i} = \frac{\partial v_k}{\partial p_i}$. Though the expression for $L_{k,i}$ is complicated, it has very nice interpretation if we use the linear approximation model (4) for the power flow and the voltage and use (1) to calculate the power loss L_k. Because the distribution line loss $L_k := r_k \ell_k = r_k \frac{P_k^2+Q_k^2}{v_k}$, we can calculate that $\frac{\partial L_k}{\partial p_i} = \frac{\partial L_k}{\partial P_k}\frac{\partial P_k}{\partial p_i} + \frac{\partial L_k}{\partial v_k}\frac{\partial v_k}{\partial p_i} = L_{k,i}$.

The trading terms: The trading terms for the bilateral transaction between an arbitrary pair of nodes can be defined by combining multiple transactions. For instance, to transfer one unit of power from node i to node j requires $\bar{\beta}_{ij}^k := \bar{\beta}_i^k - \bar{\beta}_j^k$ and $\underline{\beta}_{ij}^k := \underline{\beta}_i^k - \underline{\beta}_j^k$ units of voltage rights $\bar{v}_k$ and $\underline{v}_k$ respectively on node k, and economic rents for $\phi_{ij}^k := \phi_i^k - \phi_j^k$ units of line losses on link $(k, \pi(k))$. The market mechanism can thus be summarized as $(\bar{v}, \underline{v}, \beta, \phi)$. The distribution charge can be decomposed into two components: a rent for the voltage binding costs and a rent for the external costs associated with distribution line losses. The rent for a node voltage binding is zero when the voltage is strictly within the acceptable range and it becomes positive only when the voltage reaches its capacity, whereas the rent for line losses is nonzero most of the time, regardless of the voltage conditions.

B. Competitive equilibrium and social optimum

Let us denote by η_k the economic rent for the line losses on line $(i, \pi(i))$. Denote by η the vector of (η_i) for $i = 1, \ldots, n$. The definition of competitive equilibrium (p, λ, ξ, η) remains virtually the same as in Definition 3 with the exception that condition ii) needs to be replaced by the following version:

ii). There should be no positive profit to be made by transfer of power from one node to another. Specifically, the transaction of delivering a unit of power from node i to j requires the payment of 1) the electricity purchase price p_i 2) the voltage binding cost $\bar{\xi}_k \bar{\beta}^k_{ij} + \underline{\xi}_k \underline{\beta}^k_{ij}$ for all $k = 1, \dots, n$ 3) the cost for line losses, $\eta_k \phi^k_{ij}$. At competitive equilibrium, the net profit for such a transaction should be zero, i.e.,

$$\lambda_j = \lambda_i + \sum_{k=1}^{n} (\eta_k \phi^k_{ij} + \bar{\xi}_k \bar{\beta}^k_{ij} + \underline{\xi}_k \underline{\beta}^k_{ij}). \tag{24}$$

Theorem 2 *Under the market mechanism $(\bar{v}, \underline{v}, \beta, \phi)$, there exists such a competitive equilibrium (p, λ, ξ, μ) where p is socially optimal in the presence of line losses.*

Proof: For any p where $\underline{p}^g_i \le p^g_i \le \bar{p}^g_i$ and $\underline{p}^d_i \le p^d_i \le \bar{p}^d_i$, the Lagrangian is given by:

$$\begin{aligned}
L = & -\sum_{i=0}^{n} B_i(p^c_i) + \sum_{i=0}^{n} C_i(p^g_i) + \sum_{i=1}^{n} r_i \ell_i \\
& + \sum_{i=0}^{n} \lambda_i (P_i - \sum_{j \in \delta(i)} P_j - \sum_{j \in \delta(i)} r_i \ell_i - p^g_i + p^d_i) \\
& + \sum_{i=0}^{n} \theta_i (Q_i - \sum_{j \in \delta(i)} Q_j - \sum_{j \in \delta(i)} x_j \ell_j - q^g_i + q^d_i) \\
& + \sum_{i=1}^{n} w_i (v_i - v_{\pi(i)} - 2(r_i P_i + x_i Q_i) + (r_i^2 + x_i^2) \ell_i) \\
& + \sum_{i=1}^{n} \mu_i (\frac{P_i^2 + Q_i^2}{v_i} - \ell_i) \\
& + \sum_{i=1}^{n} \bar{\gamma}_i (v_i - \bar{v}_i) + \underline{\gamma}_i (-v_i + \underline{v}_i)
\end{aligned}$$

Thus $(p, P, Q, \ell, v, \lambda, \theta, \omega, \mu, \gamma)$ is a primal-dual optimum, if and only if, (p, P, Q, ℓ, v) are feasible, and,

$$p^d_i \in \arg \max_{p^d_i \in [\underline{p}^d_i, \bar{p}^d_i]} B_i(p^d_i) - \lambda_i p^d_i, \forall i \in N$$

$$p^g_i \in \arg \max_{p^g_i \in [\underline{p}^g_i, \bar{p}^g_i]} \lambda_i p^g_i - C_i(p^g_i), \forall i \in N$$

and for $i = 1, \dots, n$,

$$\frac{\partial L}{\partial P_i} = -\lambda_{\pi(i)} + \lambda_i + 2\mu_i \frac{P_i}{v_i} - 2 w_i r_i = 0, \tag{25}$$

$$\frac{\partial L}{\partial Q_i} = -\theta_{\pi(i)} + \theta_i + 2\mu_i \frac{Q_i}{v_i} - 2 w_i x_i = 0, \tag{26}$$

$$\frac{\partial L}{\ell_i} = r_i - \lambda_{\pi(i)} r_i - \theta_{\pi(i)} x_i - \mu_i + w_i (r_i^2 + x_i^2) = 0 \tag{27}$$

$$\frac{\partial L}{v_i} = w_i - \mu_i \frac{P_i^2 + Q_i^2}{v_i^2} - \sum_{k \in \delta(i)} w_k + \gamma_i = 0, \tag{28}$$

where $\gamma_i := \bar{\gamma}_i - \underline{\gamma}_i$ and $\bar{\gamma}_i [v_i - \bar{v}_i] = \underline{\gamma}_i [-v_i + \underline{v}_i] = 0$. For any $i = 1, \dots, n$, summing up equation (28) for all $j \in \Delta(i)$, we have,

$$w_i = \sum_{j \in \Delta(i)} \left(-\gamma_i + \mu_i \frac{P_i^2 + Q_i^2}{v_i^2} \right) \tag{29}$$

Summing up equation (25) along the path from node 0 to node i, we have,

$$\lambda_0 = \lambda_i + \sum_{j \in \mathcal{P}_i} \left(\mu_j \frac{2P_j}{v_j} - 2 r_j w_j \right). \tag{30}$$

Substituting (29) into (30), we have

$$\begin{aligned}
\lambda_0 &= \lambda_i + \sum_{j \in \mathcal{P}_i} \mu_j \frac{2P_j}{v_j} \qquad (31) \\
&\quad - \sum_{j \in \mathcal{P}_i} 2 r_j \left(\sum_{k \in \Delta(j)} \left(-\gamma_k + \mu_k \frac{P_k^2 + Q_k^2}{v_k^2} \right) \right) \\
&= \lambda_i + \sum_{k \in \mathcal{P}_i} \mu_k \frac{2P_k}{v_k} + \sum_{k=1}^{n} R_{ki} \left(\gamma_k - \mu_k \frac{P_k^2 + Q_k^2}{v_k^2} \right) \\
&= \lambda_i + \sum_{k=1}^{n} \frac{\mu_k}{r_k} L_{k,i} + \sum_{k=1}^{n} R_{ki} \bar{\gamma}_k - \sum_{k=1}^{n} 2 R_{ki} \underline{\gamma}_k
\end{aligned}$$

Defining β, ϕ according to (19, 22), and let

$$\bar{\xi}_k = \begin{cases} \frac{\sum_{j=1}^{n} R_{kj} p_j}{v_k - v_k^{nom}} \bar{\gamma}_k, & \text{if } v_k = \bar{v}_k; \\ \bar{\gamma}_k = 0, & \text{otherwise,} \end{cases} \tag{32}$$

$$\underline{\xi}_k = \begin{cases} \frac{\sum_{j=1}^{n} R_{kj} p_j}{v_k - v_k^{nom}} \underline{\gamma}_k, & \text{if } v_k = \underline{v}_k; \\ \underline{\gamma}_k = 0, & \text{otherwise,} \end{cases} \tag{33}$$

$$\eta_k = \frac{\sum_{j=1}^{n} L_{k,j} p_j}{L_k} \frac{\mu_k}{r_k}. \tag{34}$$

Then we have for any (i, j), $\lambda_j = \lambda_i + \sum_{k=1}^{n} (\eta_k \phi^k_{ij} + \bar{\xi}_k \bar{\beta}^k_{ij} + \underline{\xi}_k \underline{\beta}^k_{ij})$, which means that (p, λ, ξ, η) is a competitive equilibrium.

■

VI. A NUMERICAL EXAMPLE

To illustrate the results, we consider a three node distribution line (as shown in Figure 1) with two identical distribution lines, each of which is characterized by an inductance of $x = 3.88$ ohms and a resistance of $r = 1.6$ ohms. The nominate voltage value is assumed to be $V_0 = 12.35$ kV. The voltage constraints at node 1 and 2 is $[0.95V_0, 1.05V_0]$. The cost function assumed for each generator is in the form of $1/2ap^2 + bp$ and the utility function assumed for each consumer is in the form of $-1/2cp^2 + dp$. Table VI displays the value of the parameters for generation and consumption.[6] For expository simplicity, the reactive power injection at each node is assumed to be zero.

Under the above assumptions, the market mechanism is characterized by the issuance of $\bar{v}_k = (1.05^2 - 1)V_0^2$ and $\underline{v}_k = (1 - 0.95^2)V_0^2$ of voltage rights for each node $i = 1, 2$ and a trading rule (which is continuously updated in the

[6]Note that in order to amplify the effect of voltage constraints, we scale up the value of resistance, reactance, consumption and generation compared to a normal distribution feeder.

market trading process). As the market reaches a competitive equilibrium, the trading rule is characterized by (β, ϕ) as displayed in Table VI. The resulting competitive equilibrium is summarized in table VI.

TABLE I

PARAMETERS FOR GENERATION AND CONSUMPTION

	a	b	c	d	$[\underline{p}^g, \bar{p^g}]$	$[\underline{p}^d, \bar{p}^d]$
node 0	0.1	0.3	0.2	1.2	[0.45, 4.56]	[0.36, 0.66]
node 1	0.2	0.1	0.2	1.5	[0.06, 0.21]	[1.02, 2.52]
node 2	0.2	0.05	0.1	2.2	[0.09, 0.50]	[1.65, 2.88]

TABLE II

TRADING RULES

k	$\bar{\beta}_1^k$	$\bar{\beta}_2^k$	$\underline{\beta}_1^k$	$\underline{\beta}_2^k$	ϕ_1^k	ϕ_2^k
1	3.8377	3.8377	-3.8377	-3.8377	-0.0288	-0.0288
2	3.6625	7.3250	-3.6625	-7.3250	-0.0003	-0.0180

TABLE III

COMPETITIVE EQUILIBRIUM

k	p_k^d	p_k^g	v_k	λ_k	$\underline{\xi}_k$	$\bar{\xi}_k$	η_k
0	0.660	3.236	152.5225	0.6236			
1	1.020	0.210	143.1770	1.3432	0	0	9.1125
2	2.105	0.480	137.6516	1.9895	0.1238	0	11.5616

We can check the competitive equilibrium condition (24) by using the values provided by Table VI and Table VI. As examples, for node 0 and 1, we have $\lambda_0 = \lambda_1 + \underline{\beta}_1^2 \underline{\xi}_2 + \phi_1^1 \eta_1 + \phi_1^2 \eta_2$; for node 0 and 2, we have $\lambda_0 = \lambda_2 + \underline{\beta}_2^2 \underline{\xi}_2 + \phi_2^1 \eta_1 + \phi_2^2 \eta_2$. Note that the convex relaxation is checked to be exact for this distribution line.

VII. DISCUSSION AND CONCLUSION

In this paper, we present a market mechanism for a radial distribution network which internalizes externalities within bilateral trading transactions. We define trading rules that efficiently allocate the externalities to individual transactions. Specifically, we focus on the external costs associated with voltage constraints and distribution line losses. We show that a competitive market could be established for distribution services to achieve a social optimum.

As we have seen, the market mechanism and the analysis with distribution line losses are much more complicated than the loss-less case. We would like to further discuss the limitation of the results in this paper and the future work.

Firstly, we note that in order to ensure (20) and (23), we define a complicate trading rule for β and ϕ as in (19,22). Conditions of (20) and (23) mean that the total quantities of voltage rights and line losses requested by nodes need to equal the true voltage rights and line losses used in the network. These conditions are similar to the budge balance conditions required in other resource allocation problems [16]. If the trading rule does not need to satisfy such budget balance conditions, we can define simpler trading rules which requires less information but still guarantee the efficiency of the market mechanism. Future work involves exploring different trading rules to satisfy different requirements and information constraints.

Secondly, compared to Theorem 1, the statement in Theorem 2 is weaker. Theorem 2 guarantees that there exists a competitive equilibrium which is socially optimal but does not guarantee that every competitive equilibrium is optimal. This is because of the nonlinear power flow equations in (1). For the nonlinear case, more conditions are required to ensure a competitive equilibrium to be socially optimal.

Thirdly, we have focused on the market design and the competitive equilibrium but have not specified any trading process to reach the equilibrium as done in [10]. One way to design the trading process is to use the dual gradient algorithm to solve the social welfare maximization problem. But there are other possibilities as well, e.g., a trading process similar to the one in [10]. One future direction is to develop different trading processes, to study their convergences, and to explore their economic insights.

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