Correction: A Parallel Primal-Dual Interior-Point Method for DC Optimal Power Flow

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In our PSCC paper, the formulation we used is not equivalent to a DC-OPF problem but rather a network-flow problem. We thank our committee chair Professor Bakirtzis and Professor Chatzivasileiadis for bringing this to our attention. In this note, we update our formulation to a DC-OPF problem and demonstrate the original ideas from our PSCC still hold.

1 DC-OPF Formulation

In order for our method to work well in a distributed setting (i.e., limited communication requirements), it is important that the system for solving the Newton step has a sparse structure. At the same time, the system matrix should be reasonably conditioned so that the iterative solver is well-behaved. For these two reasons, we utilize the following formulation of DC-OPF. Consider a power network with \( n \) buses in set \( \mathcal{N} \) and \( m \) transmission lines in set \( \mathcal{E} \). Let \( \theta \in \mathbb{R}^{n-1} \) be the voltage phase angle at each bus except bus 1 (the reference bus). Let \( b_l \) be the susceptance of line \( l \) from bus \( i \) to bus \( j \), and \( b \) the vector of all line susceptances.

\[
\begin{align*}
\min_{\theta} & \quad \sum_i f_i(p_i) := f(B\theta) \\
\text{s.t.} & \quad p \leq B\theta \leq \bar{p} \\
& \quad f \leq DA\theta \leq \bar{f},
\end{align*}
\]

where the bus-susceptance \( \bar{B} \) matrix is

\[
[\bar{B}]_{ij} = \begin{cases} 
-b_i, & \text{if line } l \text{ connects buses } i \text{ and } j \\
0, & \text{if buses } i \text{ and } j \text{ are not directly connected} \\
-\sum_{j \neq i}[\bar{B}]_{ij}, & \text{if } i = j.
\end{cases}
\]

The matrix \( B \) is obtained by removing the first column from \( \bar{B} \). This results from setting the reference angle at bus 1 to 0. The line-bus incidence matrix \( \bar{A} \) to be of dimension \( m \times n \) with \( [\bar{A}]_{li} = +1 \) and \( [\bar{A}]_{lj} = -1 \) if the start bus of line \( l \) is \( i \) and the end bus of line \( l \) is \( j \). The matrix \( A \) is \( \bar{A} \) with the first column removed.

The matrix \( D \) is an \((m \times m)\)-matrix whose \( l \)th entry is \( b_l \).

In our simulations, we use a quadratic cost function of the form

\[
f(B\theta) = (B\theta - p^*)^T W (B\theta - p^*),
\]

where \( p^* \) is a vector whose entries are the nominal power demand for demand buses and zero for generation buses. The weighting matrix \( W \) is diagonal and allows to parametrize the different generator costs and demand utility functions.

2 Proposed Method and Iterative Linear Solver Convergence

We verify that our proposed method for calculating the Newton step holds under the DC-OPF formulation in (1)-(3). The Newton step requires solving
\[
\begin{bmatrix}
2B^TWB & B^T & -B^T & (DA)^T & -(DA)^T \\
-\lambda_p B & \text{diag}(B\theta - \bar{p}) & 0 & 0 & 0 \\
\lambda_p B & 0 & \text{diag}(p - B\theta) & 0 & 0 \\
-\lambda_f DA & 0 & 0 & \text{diag}(DA\theta - \bar{f}) & 0 \\
\lambda_f DA & 0 & 0 & 0 & \text{diag}(f - DA\theta)
\end{bmatrix} \\
\begin{bmatrix}
\Delta\theta \\
\Delta\lambda_p \\
\Delta\lambda_p \\
\Delta\lambda_f \\
\Delta\lambda_f
\end{bmatrix} = \\
\begin{bmatrix}
r_{\text{dual}} \\
r_{\text{cent},\bar{p}} \\
r_{\text{cent},p} \\
r_{\text{cent},\bar{f}} \\
r_{\text{cent},f}
\end{bmatrix}
\] (6)

where \(\lambda_p\) are the dual variables associated with the upper bounds on the power injections, \(B\theta - \bar{p} \leq 0\). Similar definition applies to the other dual variables. The right-hand side quantities are

\[
r_{\text{dual}} = 2B^T(WB \cdot (B\theta - p^*) + B^T(\lambda_p - \lambda_p) + (DA)^T(\lambda_f - \lambda_f))
\] (8)

\[
r_{\text{cent},\bar{p}} \overset{\text{def}}{=} [\lambda_p(B\theta - \bar{p}) - \frac{1}{1}1] \\
r_{\text{cent},p} \overset{\text{def}}{=} [\lambda_p(p - B\theta) - \frac{1}{1}1] \\
r_{\text{cent},\bar{f}} \overset{\text{def}}{=} [\lambda_p(DA\theta - \bar{f}) - \frac{1}{1}1] \\
r_{\text{cent},f} \overset{\text{def}}{=} [\lambda_p(f - DA\theta) - \frac{1}{1}1]
\] (9)

First, we eliminate the dual variables. Since the equality constraints are linear and separable in \(\theta\), this is computationally inexpensive and maintains sparsity of the system. Grouping all of the inequality constraints from (2)-(3) into \(g(\theta) \leq 0\), let

\[
g(\theta) \overset{\text{def}}{=} \begin{bmatrix} B\theta - \bar{p} \\ p - B\theta \\ DA\theta - \bar{f} \\ f - DA\theta \end{bmatrix}, \quad Dg(\theta) = \begin{bmatrix} B \\ -B \\ DA \\ -DA \end{bmatrix}
\] (10)

The reduced system is

\[
C\Delta\theta = w,
\] (11)

where the system matrix and right-hand side are

\[
C = 2B^TWB + Dg(\theta)^T\text{diag}(-g(\theta))^{-1}\text{diag}(\lambda)Dg(\theta)
\] (12)

\[
w = -r_{\text{dual}} - Dg(\theta)^T\text{diag}(g(\theta))^{-1}r_{\text{cent}}.
\] (13)

Since \(\text{diag}(g(\theta))\) is diagonal, \(M\) exhibits a sparse structure. The original matrix-splitting scheme can be applied to \(C\) in (12). Consider decomposing the matrix \(C\) into the difference of two matrices \(M\) and \(N\). Provided the spectral radius \(\rho(M^{-1}N)\) is less than one, let the fixed-point of the sequence

\[
\Delta\theta^{t+1} = M^{-1}N\Delta\theta^t + M^{-1}w,
\] (14)
be $\Delta \theta^*$, then $C \Delta \theta^* = w$. The splitting (i.e. choice of $M$ and $N$) should be designed so that i) the matrix-splitting iterates converge and ii) the iterates in (14) are easy to calculate in a distributed way.

Without loss of generality, assume the bus ids and line ids are consecutively assigned across control areas. The matrix $C$ can be decomposed into the sum of a block-diagonal matrix, $D$, and a matrix containing the remaining off-diagonal entries $E$. The consecutive ordering of the bus and line indices described above allows for all entries of $C$ corresponding to buses within the same control area to be contained within a single diagonal block. Specifically, let

$$D_{ij} = \begin{cases} 
C_{ij} & \text{if buses } i \text{ and } j \text{ belong to the same control area}, \\
0 & \text{otherwise}
\end{cases} \quad (15)$$

$$E_{ij} = \begin{cases} 
C_{ij} & \text{if buses } i \text{ and } j \text{ belong to different control areas}, \\
0 & \text{otherwise}
\end{cases} \quad (16)$$

yielding $C = D + E$. We use the following matrix-splitting design

$$M = D + \tau \bar{E}, \quad N = \tau \bar{E} - E, \quad (17)$$

where $\tau$ is a scalar parameter and $\bar{E}$ is a diagonal matrix whose $i$th diagonal entry equals

$$\bar{E}_{ii} \eqdef \sum_{j \neq i} |A_{ij}|. \quad (18)$$

This is a block-Jacobi scheme modified to be diagonally dominant. We have the following proposition which ensures convergence of the matrix-splitting iterates in (14).

**Proposition 1.** Using the splitting in (17), for $\tau \geq \frac{1}{2}$, the iterative updates in (14) converge.

**Proof.** The sequence $\{\Delta v^t\}$ in (14) converges to its limit $\Delta v^*$ as $t \to \infty$ if and only if the spectral radius of the matrix $M^{-1}N$ is strictly less than 1 [1]. Furthermore, if the sequence converges, the limit $\Delta \theta^*$ is the solution of the system, (i.e., $C \Delta \theta^* = w$). In order to have the spectral radius $\rho(M^{-1}N) < 1$, it is sufficient to have $C = M - N \succ 0$ and $M + N \succ 0$ [2].

To show that $C \succ 0$, first we note that the Hessian of the objective function $f_0(\theta) \eqdef (B\theta - p^*)^T W (B\theta - p^*)$ is diagonal and assumed to have strictly positive entries. Then, it is sufficient to show that 1) $Dg(\theta)$ has full column rank and 2) $\text{diag}(-g(\theta))^{-1}\text{diag}(\lambda)$ has strictly positive entries. Note from (10) that $Dg(\theta)$ has full column rank by construction since $B$ is the bus susceptance matrix with the column corresponding to the reference angle eliminated. For each constraint $i$ the entries

$$[\text{diag}(g(\theta))]^{-1}\text{diag}(\lambda)|_{ii} = \frac{\lambda_i}{-g_i(\theta)} > 0 \quad (19)$$

since $\lambda_i > 0$ and $g_i(\theta) < 0$. Lastly, $M + N$ is positive definite due to its construction which makes it diagonally dominant. Details on this are provided in [3].

### 3 Updated Results

In Figure 3 (a)-(d), we show updated results using the new formulation. The IEEE 14-bus system is used in (a) and (b), and the IEEE 118-bus system is used in (c) and (d). The convergence of the relative error of the objective function for the centralized PDIP algorithm (i.e., using direct inversion to calculate the Newton step) is compared to the distributed method (i.e., using the proposed matrix-splitting method to calculate the Newton step). A tolerance of $10^{-10}$ is set to terminate the matrix-splitting iterates. For 118-bus system, between $10^5$ and $10^6$ matrix-splitting iterates are required per iteration. This is larger than in the original results using a simpler network-flow problem. While the number of inner-loop iterations required is high, each inner-loop iteration is a computationally inexpensive arithmetic calculation. Future work will be done to study the trade-off between the inner-loop tolerance and the outer-loop convergence. To study the quality of the solution, the convergence of the residuals using our distributed method is shown in (b) and (d) for the 14-bus and 118-bus systems, respectively.
(a) Convergence of Obj. Function Relative Error: 14-Bus System  
(b) Convergence of Residuals: 14-Bus Study

(c) Convergence of Obj. Function Relative Error: 118-Bus System  
(d) Convergence of Residuals: 118-Bus Study

References

