Distributed Control for Reaching Optimal Steady State in Network Systems: An Optimization Approach

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***Abstract*—In this paper, we consider the problem of distributed control for network systems to achieve optimal steady-state performance. Motivated by recent research on re-engineering cyber-physical systems such as power systems and the Internet, we propose a two-step control retrofit procedure. Firstly, we reformulate the dynamical system as an optimization algorithm to solve a certain optimization problem. Secondly, we combine a predefined steady-state optimization problem and the reformulated problem to systematically (re)design the control. As a result, the system automatically tracks the optimal solution of the predefined steady-state optimization problem and the control scheme can be implemented in a distributed and closed-loop manner. In order to investigate how general this framework is, we establish necessary and sufficient conditions under which a linear dynamical system can be viewed as an optimization algorithm. These conditions are characterized using properties of system matrices and related linear matrix inequalities. Lastly, a practical example of frequency control in power systems is provided to show the effectiveness of the proposed framework.**

***Index Terms*—Distributed control, electric power systems, gradient algorithms, reverse-engineering, control retrofit.**

I. Introduction

In recent years, advanced communication, computing, sensing, and actuation technologies have been stimulating the emergence of network systems, such as the smart grid [2], smart buildings [3], mobile robots [4], and intelligent transportation systems [5]. These networks usually operate in an uncertain environment and with incomplete information. To ensure their stability and economic efficiency, they usually operate at two different time-scales. At a slow time-scale, efficient nominal operating points are determined using optimization methods with predictions of future uncertainties. At a fast time-scale (in real-time), the nominal points are tracked and the system is stabilized using control techniques. However, as the uncertainties fluctuate faster and by a larger amount, this time-scale separation framework tends to induce a large economic loss and even cause instability. For example, there are faster and larger fluctuations in power systems, because of the increasing penetration of renewable energy sources such as solar and wind power. As a result, conventional frequency control schemes become much less economically efficient [6].

The goal of this paper is to (re)design (existing) distributed control mechanisms[1] so that the network system automatically tracks an efficient operating point, i.e., the system can converge to an optimal steady state which is defined by a predefined optimization problem no matter what the disturbance is. By doing this, the gap between the aforementioned two time-scales is bridged, and there is no need to compute the nominal operating point for the system beforehand.

Distributed control for network systems has been actively studied for several decades [7]–[13]. However, an open question remains: how to design distributed control (e.g. H_2, H_∞, linear quadratic control, model predictive control) to satisfy communication constraints. This is in general NP-hard and intractable, even for the case of Linear Time-Invariant (LTI) systems [14]. The major difference of the problem studied in this paper is that the literature aims to design a distributed control scheme to optimize the system's dynamic behavior over an entire time horizon (also the system may have a special structure, e.g., a positive system); whereas we aim to optimize the steady-state behavior[2]. This allows us to derive tractable distributed (or even decentralized) solutions using an optimization approach. In fact, the idea of using optimization methods to do control design can be traced back to the works of Pontryagin, in which gradient descent type algorithms were used to track some implicitly defined optimal point (furthermore, the assumptions of convexity and Slater's condition yield global optima). Nowadays, there is a growing interest in cyber-physical systems that studies distributed control via an optimization view, such as frequency control in power systems and congestion control for the Internet, [15]–[19]. Most of these approaches connect the system dynamics with optimization algorithms, and use this connection to design distributed control for achieving optimal steady-state performance. This technical note provides a generalized version of those earlier

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[1] "Redesign" and "existing" refer to the case in which the system is already equipped with some existing built-in control, e.g., automatic generation control in power grids. In this case, the system designer may only plan to modify the existing control instead of designing a new control that might be very different from the built-in control, as it is more costly to rebuild a new controller.

[2] By "steady-state" behavior, we have an underlying assumption that the disturbance is a step change, i.e., it is constant/slow varying. This can simplify the proofs in the paper while the controller can also be applied to the time-varying disturbance case, as will be shown in Section V.

works, focusing on how to extend the control design procedure to general linear dynamical systems and what type of systems this design procedure can be applied to.

Motivated by the aforementioned successes, we propose and study a systematic approach to (re)design (existing) distributed control schemes for reaching an optimal steady state (with respect to a predefined optimization problem) in linear network systems. Firstly, we reformulate a dynamical system as a gradient algorithm to solve an optimization problem. Secondly, we use a retrofit approach to systematically design the control structure or modify the existing control mechanism. As a result, the redesigned system can automatically track the optimal solution of the predefined optimization problem. Moreover, the resulting controller (i) has a distributed/decentralized structure and can be implemented in a closed-loop manner (i.e., no information of disturbances is needed); (ii) respects the operating constraints; (iii) ensures an efficient and reliable network operation. In order to investigate how general this approach is, we establish necessary and sufficient conditions under which a linear dynamical system can be viewed as an optimization algorithm. These conditions are characterized using properties of the system matrices and related linear matrix inequalities.

The remainder of this paper is organized as follows. In Section II, we provide the problem formulation. In Section III, we present the retrofit procedure to (re)design distributed control mechanisms for a special class of network systems. In Section IV, we develop necessary and sufficient conditions for distinguishing this special class of systems. Section V provides a numerical example to show the effectiveness of our approach, and Section VI concludes the paper.

A. Notation

$\dot{x}(t)$ denotes the derivative of a state variable $x(t)$ with respect to time t, i.e., $\dot{x}(t) = \frac{d}{dt}x(t)$. $x \in \mathbb{R}^m$ is a column vector in an m-dimensional Euclidean space $\mathbb{R}^m$. $X \in \mathbb{R}^{m\times n}$ is an $m \times n$ real matrix. diag$\{\star\}$ is a diagonal matrix with corresponding entries $\star$ on the main diagonal. x^T (x^H) is the transpose (conjugate transpose) of x. I_m is an identity matrix of size $m \times m$. $\mathbf{0}$ is a matrix of zeros of appropriate dimension determined by context. eig(X) is the spectrum of a square matrix X. $X \succeq 0$ ($X \succ 0$) denotes that a square matrix X is positive semi-definite (positive definite). $\nabla_x f(x)$ or $\frac{\partial f}{\partial x}$ is the gradient (as a column vector) of a scalar function $f \in \mathcal{C}^1$ of x, where $\mathcal{C}^n$ is the class of functions that are n times continuously differentiable. $\nabla_x^2 f(x)$ is the Hessian matrix of a scalar function $f \in \mathcal{C}^2$ of x. The positive projection of a function $h(y)$ on a variable x, $(h(y))_x^+$ is:

$$(h(y))_x^+ = \begin{cases} h(y) & \text{if } x > 0 \\ \max(0, h(y)) & \text{if } x = 0 \end{cases}.$$

II. Problem Setup and Preliminaries

A. Problem Setup

Consider a network system consisting of N subsystems

$$\dot{x}_i = \sum_{j\in\mathcal{N}(i)} A_{ij}x_j + B_i u_i + C_i w_i,\ i = 1,\cdots,N. \quad (1)$$

Each subsystem is equipped with a *built-in* controller

$$\dot{u}_i = \sum_{j\in\mathcal{N}(i)} D_{ij}x_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_i w_i, i = 1,\cdots,N \quad (2)$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state vector of subsystem i, $\mathcal{N}(i)$ is the set of neighboring subsystems of subsystem i, $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $B_i \in \mathbb{R}^{n_i\times m_i}$, $u_i(t) \in \mathbb{R}^{m_i}$ is the control input vector to subsystem i, $C_i \in \mathbb{R}^{n_i\times p_i}$, $w_i(t) \in \mathbb{R}^{p_i}$ is the exogenous input vector to subsystem i, e.g., disturbance injection, $D_{ij} \in \mathbb{R}^{m_i\times n_j}$, $E_{ij} \in \mathbb{R}^{m_i\times m_j}$ and $F_i \in \mathbb{R}^{m_i\times p_i}$. Here the dynamic feedback controller (2) is assumed to have a distributed structure. For convenience, let $x = \{x_1,\cdots,x_N\}$, $u = \{u_1,\cdots,u_N\}$, $w = \{w_1,\cdots,w_N\}$, $n = \sum_{i=1}^N n_i$, $m = \sum_{i=1}^N m_i$ and $p = \sum_{i=1}^N p_i$.

Remark 1. *System (1)-(2) describes a general class of network systems with built-in control mechanisms. Equation (2) represents any distributed dynamic feedback controller of order one or more. Moreover,* $D_{ij} = \mathbf{0}, E_{ij} = \mathbf{0}, F_i = \mathbf{0}, \forall j \in \mathcal{N}(i)$ *means that there is no such controller equipped with subsystem i. Note that a zero-order distributed state feedback controller, i.e.,* $u_i = \sum_{j\in\mathcal{N}(i)} K_{ij}x_j$ *where* $K_{ij} \in \mathbb{R}^{m_i\times n_j}$*, can be included in the system dynamics (1) through* A_{ij}*. If* $F_i = \mathbf{0}$*, then the built-in controller of subsystem* i *does not use any information on the disturbance* w_i*. Lastly, if the considered network system has nonlinear dynamics with built-in (nonlinear) control, system (1)-(2) can be obtained through linearization around an operating point/equilibrium point, and therefore indicates local behaviour of the nonlinear system. In this case, all results proposed in this paper become local.*

Suppose that the exogenous input w is a step disturbance, and that the closed-loop system (1)-(2) is stabilized. It is straightforward to see that the trajectory converges to one point in the equilibrium set $\mathcal{X} := \{x | \sum_{j\in\mathcal{N}(i)} A_{ij}x_j + B_i u_i + C_i w_i = \mathbf{0}, \sum_{j\in\mathcal{N}(i)} D_{ij}x_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_i w_i = \mathbf{0}, i = 1,\cdots,N\}$ (in practice, there can be multiple equilibria/a set of equilibria for (1)-(2), e.g., when adopting a proportional-integral or an integral controller for frequency control in power grids [20]). Now consider the following optimization problem associated with the system under a constant disturbance w:

$$\min_{y\in\mathbb{R}^n, u\in\mathbb{R}^m} \sum_{i=1}^N f_i(y_i) + \sum_{i=1}^N g_i(u_i) \quad (3a)$$

$$\text{s.t.} \sum_{j\in\mathcal{N}(i)} A_{ij}y_j + B_i u_i + C_i w_i = \mathbf{0} \quad (3b)$$

$$\sum_{j\in\mathcal{N}(i)} D_{ij}y_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_i w_i = \mathbf{0} \quad (3c)$$

$$h_i(y_i, u_i, y_j, u_j) \le 0, j \in \mathcal{N}(i) \quad (3d)$$

where $i = 1,\cdots,N$, y (y_i) denotes the expected steady state (here we replace x (x_i) with y (y_i) to distinguish between the system states and the steady-state values), $f_i \in \mathcal{C}^2$: $\mathbb{R}^{n_i} \to \mathbb{R}$, $g_i \in \mathcal{C}^2$: $\mathbb{R}^{m_i} \to \mathbb{R}$, $h_i \in \mathcal{C}^2$: $\mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{\sum_{j\in\mathcal{N}(i)} n_j} \times \mathbb{R}^{\sum_{j\in\mathcal{N}(i)} m_j} \to \mathbb{R}$, and the objective function (3a) is assumed to be in a separable form. We assume that (3) is convex (i.e., $\nabla_{y_i}^2 f_i \succeq 0$, $\nabla_{u_i}^2 g_i \succeq 0$ and $\nabla_{y_i,u_i,y_j,u_j}^2 h_i \succeq 0$ hold), feasible and satisfies Slater's constraint qualification [21]. This optimization problem usually defines a desired operating point for the network system. For instance, in power systems, (3) may be an Optimal Power Flow (OPF) problem in which the generation cost is minimized and the load utility is maximized subject to power flow balance constraints and network operating constraints [22]. Another example is the space satellite formation control in which (3) is usually formulated as a consensus problem with respect to the positions and velocities of space satellites [23].

Remark 2. *In (3), there can be multiple inequality constraints for each subsystem, i.e., Equation (3d) can be reformulated in a vector form. For convenience, in this paper, we consider the case in which there is only one inequality constraint for each subsystem, while the result can be immediately extended to the case of multiple inequalities.*

Our goal is to redesign the existing controller (2) in a distributed and closed-loop manner so that the overall system can track the optimal solution of (3) automatically. Mathematically speaking, we will design and introduce a modification u_i^{new} to the controller (2) in the following way

$$\dot{u}_i = \sum_{j\in\mathcal{N}(i)} D_{ij}x_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_i w_i + u_i^{\text{new}} \tag{4}$$

such that the redesigned controller (4) drives (stabilizes) system (1) to a steady-state which is an optimal solution to the optimization problem (3). The additional requirement on u_i^{new} is that the updating rule of u_i^{new} does not use any information on the disturbance w (*closed-loop*) and it only depends on locally available information (*distributed*): local state information $\{x_j, u_j\}_{j\in\mathcal{N}(i)}$ and/or other local auxiliary variables which are introduced to assist the computation[3]. Note that the original built-in control (2) is preserved in the new controller (4) and the system dynamics (1) are kept intact. This property makes the controller (4) easier to implement as only an additional control command u_i^{new} needs to be introduced.

Remark 3. *The reason to choose the problem setup as in (1)-(2) and (4) is as follow. In practice, there are usually existing controllers incorporated (e.g., PI-type controllers) in the system. In this case, the system designer may only plan to modify the existing control law instead of designing a new controller to save on the implementation cost. Mathematically, this puts a design constraint on where we can add/implement the new part of the controller, e.g., in Equation (4). On the other hand, assuming the nominal system is stable and then aiming to design distributed control would be a different problem, which has been studied in, e.g., [24]–[26]. Our retrofit procedure can be applied to solve the problem under this setup but not the other way around.*

As we discussed before, in the existing literature and practice, e.g., frequency control in power systems, firstly a nominal operating point is computed by solving some optimization problem like (3) *using a prediction of the future disturbance* w and then a controller (2) is designed to track this nominal point. Moreover, the optimization problem is usually solved (i) at a much slower time-scale compared to system dynamics (1)-(2) and (ii) by using centralized optimization or distributed optimization methods, both of which require a certain amount of communication and computation. If controller (4) can be designed to track the optimal solution of (3) automatically, then the system itself can adapt to the changing and uncertain disturbance w. Moreover, the overhead of communication and computation that are used to solve (3) can be reduced.

Assumption 1. *In this paper, we assume that (3) is convex, feasible and satisfies Slater's condition. Moreover, we assume that the unknown disturbance w is constant or slow varying to carry out the control design and analysis (recall that the update of (2) can indirectly rely on w without exactly measuring it). However, as our designed controller u_i^{new} does not need any information of w, the controller works under time-varying w. In the numerical study of Section V, we also test the performance of the controller under time-varying disturbances. We leave it as future work to analyze the performance of the proposed controller under time-varying w.*

B. Preliminaries

In this paper, we focus on a special class of systems (1)-(2), which can be interpreted as a primal-dual gradient algorithm [16], [27] to solve an unconstrained quadratic saddle point problem. Given a function $f \in \mathcal{C}^2$: $\mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}$, $(\tilde{y}, \tilde{z})$ is a saddle point of f if $f(\tilde{y}, z) \le f(\tilde{y}, \tilde{z}) \le f(y, \tilde{z})$ for all y, z (we abuse the notation y, while its meaning should be clear from the context). Assume that for all y, z, $\nabla_y^2 f \succeq 0, \nabla_z^2 f \preceq 0$, and the set $\{(y,z)|\nabla_{y,z} f = \mathbf{0}\}$ is nonempty. Then a primal-dual gradient algorithm to solve $\max_z \min_y f(y,z)$ is:

$$\dot{y} = -K_y \frac{\partial f}{\partial y}, \quad \dot{z} = K_z \frac{\partial f}{\partial z} \tag{5}$$

where $K_y \in \mathbb{R}^{a\times a}, K_z \in \mathbb{R}^{b\times b}$ are positive definite constant matrices. The following lemma shows the convergence of (5).

Lemma 1. *Suppose for all y, z, $\nabla_y^2 f \succeq 0, \nabla_z^2 f \preceq 0$, and the set $\{(y,z)|\nabla_{y,z} f = \mathbf{0}\}$ is nonempty. Then the trajectories of the primal-dual gradient dynamics/saddle point dynamics (5) are bounded. Furthermore, if f is either strictly convex in y or strictly concave in z, each trajectory of (5) asymptotically converges to a saddle point of f.*

Proof. See the proof of Lemma 5.2 in [28] (pp. 140–141). □

Formally speaking, we focus on a network system (1)-(2) which belongs to the following class:

Class-$\mathcal{S}'$[4]: System (1)-(2) belongs to Class-$\mathcal{S}'$ if there exists a function $L_{\text{sys}}(x^{(1)}, x^{(2)}, u) : \mathbb{R}^{m+n} \to \mathbb{R}$ and positive definite matrices $P_{x^{(1)}}$, $P_{x^{(2)}}$ and $P_{u_i}, i = 1, \cdots, N$ such that $\nabla^2_{x^{(1)}} L_{\text{sys}} \preceq 0$, $\nabla^2_{x^{(2)},u} L_{\text{sys}} \succeq 0$, the saddle point set $\{(x^{(1)}, x^{(2)}, u)|\nabla_{x^{(1)},x^{(2)},u} L_{\text{sys}} = \mathbf{0}\}$ is nonempty, and system (1)-(2) is a primal-dual gradient algorithm to solve $\max_{x^{(1)}} \min_{x^{(2)},u} L_{\text{sys}}$, i.e., $\dot{x} = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}\}\frac{\partial L_{\text{sys}}}{\partial x}$ and $\dot{u}_i = -P_{u_i}\frac{\partial L_{\text{sys}}}{\partial u_i},\ i = 1, \cdots, N$.

[3]A concrete form of u_i^{new} is given in (8).

[4]Notation $\mathcal{S}$ stands for *saddle point dynamics*.

Remark 4. *In the above definition, (i) state x is partitioned into $x^{(1)}$ and $x^{(2)}$; (ii) u appears as a minimizer; (iii) the gain matrix for $\dot{u}$ is block diagonal consisting of $P_{u_i}, i = 1, \cdots, N$, i.e, $\dot{u}$ has a distributed structure. The definition can be extended to the case where u appears as a maximizer by adding a minus sign before L_{sys}. For ease of exposition, we only focus on Class-$\mathcal{S}'$ in the above definition.*

The motivation to consider this special class is mainly because: (i) recent research has demonstrated that there are many cyber-physical systems, such as power system frequency control and Internet congestion control, belonging to this class [17], [18]; (ii) if a system belongs to Class-$\mathcal{S}'$, then we have a systematic procedure from an optimization view to redesign the controller (2) for achieving the optimal steady-state performance in (3), as demonstrated in Section III.

Although recent work has shown that power system frequency control dynamics and Internet congestion control protocols belong to Class-$\mathcal{S}'$, the motivation of this paper is to investigate *how general this class is* so that the aforementioned systematic approach can be applied to a broad range of network systems. This leads to Section IV in which we provide *necessary and sufficient conditions* under which the system dynamics (1)-(2) can be interpreted as a primal-dual gradient algorithm. Before that, we would like to elaborate on the systematic approach to show the nice property Class-$\mathcal{S}'$.

III. Control Retrofit from an Optimization View

In this section we provide a procedure to redesign the controller of system (1)-(2) so that the new closed-loop system can track the optimal solution of (3) automatically, under the premise that system (1)-(2) belongs to Class-$\mathcal{S}'$.

Step 1): Merge optimization problems
Since problem (3) is convex and strong duality holds, derive a saddle point problem corresponding to (3), given by

$$\begin{aligned}\max_{\zeta_i\in\mathbb{R}^{n_i},\lambda_i\in\mathbb{R}^{m_i},\mu_i\geq 0}\ \min_{y\in\mathbb{R}^n,u\in\mathbb{R}^m} L_{\text{op}} = \sum_{i=1}^N f_i(y_i) + \sum_{i=1}^N g_i(u_i)\\ + \sum_{i=1}^N \mu_i h_i(y_i,u_i,y_j,u_j) - \sum_{i=1}^N \zeta_i^T\Bigg(\sum_{j\in\mathcal{N}(i)} A_{ij}y_j + B_i u_i\\ + C_i w_i\Bigg) - \sum_{i=1}^N \lambda_i^T\Bigg(\sum_{j\in\mathcal{N}(i)} D_{ij}y_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_i w_i\Bigg)\end{aligned} \tag{6}$$

where $\zeta_i \in \mathbb{R}^{n_i}, \lambda_i \in \mathbb{R}^{m_i}, \mu_i \geq 0$ are Lagrangian multipliers (dual variables) for the constraints in (3). By adding L_{sys}, we obtain an augmented saddle point problem:

$$\begin{aligned}\max_{\zeta_i\in\mathbb{R}^{n_i},\lambda_i\in\mathbb{R}^{m_i},\mu_i\geq 0,x^{(1)}\in\mathbb{R}^{n^{(1)}}}\ \min_{y\in\mathbb{R}^n,u\in\mathbb{R}^m,x^{(2)}\in\mathbb{R}^{n^{(2)}}} L_{\text{au}} = L_{\text{sys}}\\ + \gamma L_{\text{op}}\end{aligned} \tag{7}$$

where $n^{(1)} + n^{(2)} = n$ and $\gamma > 0$. The next lemma shows the properties of L_{au}.

Lemma 2. *$\nabla^2_{y,u,x^{(2)}} L_{au} \succeq 0$ and $\nabla^2_{\zeta_i,\lambda_i,\mu_i,x^{(1)}} L_{au} \preceq 0$ hold. Moreover, if A is invertible in (1) (here we rewrite (1) as $\dot{x} = Ax + Bu + Cw$, where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{n\times p}$), then $(y, u, x, \zeta_i, \lambda_i, \mu_i)$ is a saddle point of L_{au} if and only if $(y, u, \zeta_i, \lambda_i, \mu_i)$ is a saddle point of L_{op} and (x, u) is a saddle point of L_{sys}.*

Proof. See the proof of Lemma 5.3 in [28] (pp. 111–112). □

Step 2): Forward-engineering
With the new saddle point function L_{au}, derive the following saddle point dynamics:

$$\dot{x}_i = \sum_{j\in\mathcal{N}(i)} A_{ij}x_j + B_i u_i + C_i w_i \tag{8a}$$

$$\begin{aligned}\dot{u}_i = \sum_{j\in\mathcal{N}(i)} D_{ij}x_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_i w_i - \gamma P_{u_i}\Bigg(\frac{\partial g_i}{\partial u_i}\\ - B_i^T\zeta_i - \sum_{j\in\mathcal{N}(i)} E_{ji}^T\lambda_j + \mu_i\frac{\partial h_i}{\partial u_i} + \sum_{j\in\mathcal{N}(i)}\mu_j\frac{\partial h_j}{\partial u_i}\\ + K_{eu_i}(u_i - \hat{u}_i)\Bigg)\end{aligned} \tag{8b}$$

$$\dot{\hat{u}}_i = \hat{K}_{eu_i}(u_i - \hat{u}_i) \tag{8c}$$

$$\begin{aligned}\dot{y}_i = -K_{y_i}\Bigg(\frac{\partial f_i}{\partial y_i} - \sum_{j\in\mathcal{N}(i)} A_{ji}^T\zeta_j - \sum_{j\in\mathcal{N}(i)} D_{ji}^T\lambda_j + \mu_i\frac{\partial h_i}{\partial y_i}\\ + \sum_{j\in\mathcal{N}(i)}\mu_j\frac{\partial h_j}{\partial y_i} + K_{ey_i}(y_i - \hat{y}_i)\Bigg)\end{aligned} \tag{8d}$$

$$\dot{\hat{y}}_i = \hat{K}_{ey_i}(y_i - \hat{y}_i) \tag{8e}$$

$$\dot{\zeta}_i = -K_{\zeta_i}\Bigg(\sum_{j\in\mathcal{N}(i)} A_{ij}y_j + B_i u_i + C_i w_i + K_{e\zeta_i}(\zeta_i - \hat{\zeta}_i)\Bigg) \tag{8f}$$

$$\dot{\hat{\zeta}}_i = \hat{K}_{e\zeta_i}(\zeta_i - \hat{\zeta}_i) \tag{8g}$$

$$\begin{aligned}\dot{\lambda}_i = -K_{\lambda_i}\Bigg(\sum_{j\in\mathcal{N}(i)} D_{ij}y_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_i w_i\\ + K_{e\lambda_i}(\lambda_i - \hat{\lambda}_i)\Bigg)\end{aligned} \tag{8h}$$

$$\dot{\hat{\lambda}}_i = \hat{K}_{e\lambda_i}(\lambda_i - \hat{\lambda}_i) \tag{8i}$$

$$\dot{\mu}_i = k_{\mu_i}(h_i(y_i,u_i,y_j,u_j))^+_{\mu_i} \tag{8j}$$

where $K_{eu_i}, \hat{K}_{eu_i}, K_{\lambda_i}, K_{e\lambda_i}, \hat{K}_{e\lambda_i} \in \mathbb{R}^{m_i\times m_i}$, $K_{y_i}, K_{ey_i}, \hat{K}_{ey_i}, K_{\zeta_i}, K_{e\zeta_i}, \hat{K}_{e\zeta_i} \in \mathbb{R}^{n_i\times n_i}$ are constant and positive definite diagonal matrices, $k_{\mu_i} > 0$, and $i = 1, \cdots, N$. Note that the original built-in control is preserved after the redesign, as shown in the dynamics of $\dot{u}_i$ (Equations (2) and (8b)).

The above saddle point algorithm is not exactly the primal-dual gradient algorithm (5). It is a modified saddle point algorithm given by the following lemma.

Lemma 3. *Let $f \in \mathcal{C}^2$: $\mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}$ satisfy: for all y, z, $\nabla^2_y f \succeq 0, \nabla^2_z f \preceq 0$, and the set $\{(y,z)|\nabla_{y,z} f = \mathbf{0}\}$ is nonempty. Then the following modified primal-dual gradient algorithm*

$$\dot{y} = -K_y\left(\frac{\partial f}{\partial y} + K_{ey}(y - \hat{y})\right), \quad \dot{\hat{y}} = \hat{K}_{ey}(y - \hat{y}) \tag{9a}$$

$$\dot{z} = K_z \left(\frac{\partial f}{\partial z} - K_{ez}(z - \hat{z}) \right), \quad \dot{\hat{z}} = \hat{K}_{ez}(z - \hat{z}) \tag{9b}$$

asymptotically converges to an equilibrium point which is a saddle point of f. Here $\hat{y}(t) \in \mathbb{R}^a, \hat{z}(t) \in \mathbb{R}^b$ are auxiliary state vectors, and $K_{ey}, \hat{K}_{ey} \in \mathbb{R}^{a \times a}, K_{ez}, \hat{K}_{ez} \in \mathbb{R}^{b \times b}$ are constant and positive definite diagonal matrices.

Proof. See the proof of Lemma 5.4 in [28] (pp. 113–114). □

Remark 5. *The reason to use algorithm (9) in the redesign is that it can always ensure pointwise convergence of the closed-loop system, even when the objective function (3a) is not strongly convex. Note that system (8) is slightly different from (9), in which only $u_i, y_i, \zeta_i, \lambda_i$ are equipped with auxiliary decision variables. Also, the positive projection in (8j) does not affect the convergence property of the dynamics [27]. Similar examples are shown in [29], [30].*

Step 3): Closed-loop implementation
Let $\tilde{\zeta}_i = x_i + K_{\zeta_i}^{-1}\zeta_i$ and $\tilde{\lambda}_i = u_i + K_{\lambda_i}^{-1}\lambda_i$. Rewrite Equations (8f)-(8i) as (also substituting $\zeta_i = K_{\zeta_i}(\tilde{\zeta}_i - x_i), \lambda_i = K_{\lambda_i}(\tilde{\lambda}_i - u_i)$ into Equations (8b) and (8d))

$$\dot{\tilde{\zeta}}_i = \sum_{j \in \mathcal{N}(i)} A_{ij}(x_j - y_j) - K_{e\zeta_i}(K_{\zeta_i}(\tilde{\zeta}_i - x_i) - \hat{\zeta}_i) \tag{10a}$$

$$\dot{\hat{\zeta}}_i = \hat{K}_{e\zeta_i}(K_{\zeta_i}(\tilde{\zeta}_i - x_i) - \hat{\zeta}_i) \tag{10b}$$

$$\begin{aligned} \dot{\tilde{\lambda}}_i = & \sum_{j \in \mathcal{N}(i)} D_{ij}(x_j - y_j) - \gamma P_{u_i} \Bigg(\frac{\partial g_i}{\partial u_i} - B_i^T K_{\zeta_i}(\tilde{\zeta}_i - x_i) \\ & - \sum_{j \in \mathcal{N}(i)} E_{ji}^T K_{\lambda_j}(\tilde{\lambda}_j - u_j) + \mu_i \frac{\partial h_i}{\partial u_i} + \sum_{j \in \mathcal{N}(i)} \mu_j \frac{\partial h_j}{\partial u_i} \\ & + K_{eu_i}(u_i - \hat{u}_i) \Bigg) - K_{e\lambda_i}(K_{\lambda_i}(\tilde{\lambda}_i - u_i) - \hat{\lambda}_i) \end{aligned} \tag{10c}$$

$$\dot{\hat{\lambda}}_i = \hat{K}_{e\lambda_i}(K_{\lambda_i}(\tilde{\lambda}_i - u_i) - \hat{\lambda}_i) \tag{10d}$$

so that the extra states $\tilde{\zeta}_i, \tilde{\lambda}_i$ are independent of w.

For the optimality and stability of system (8), we have the following theorem.

Theorem 1. *If (1)-(2) belongs to Class-$\mathcal{S}'$ and A has no eigenvalues on the imaginary axis (recall Lemma 2 for the definition of A), each trajectory of (8) asymptotically converges to an equilibrium point at which (x, u) is an optimal solution of (3).*

Proof. See the Appendix. □

Remark 6. *Because the dynamics of the system and the controller are primal-dual gradient dynamics of a saddle point problem, the convergence rate follows the standard literature on primal-dual gradient algorithms [27], [31]–[35]. The convergence rate depends on the smoothness and convexity of the functions f, g, h. For instance, when they are smooth and strongly convex, the system converges to the steady state exponentially.*

Going from the original closed-loop system (1)-(2) to the one with the modified controller (8), we have introduced extra dynamics while the structure of the original dynamic feedback controller is preserved as shown in Equation (8b). The benefit of this control retrofit procedure is summarized as follows. First, the modification allows us to embed different types of steady-state convex optimization problems. Second, as long as the steady-state optimization problem is decomposable, e.g., with separable objective functions and local constraints as shown in problem (3), the resulting extra control dynamics are completely distributed as given in system (8). Third, the modification ensures that the closed-loop system can achieve optimal steady-state performance without any information of the disturbance w, i.e., the system itself can adapt to changes in the objective optimization problem (3). Thus the controller can be implemented even when the disturbance is time-varying. We leave the performance analysis under time-varying disturbances for future research.

IV. LTI Systems as Gradient Algorithms for Quadratic Optimization

In the previous section, a systematic approach has been provided to (re)design distributed control for a certain type of systems to achieve an optimal steady state. Class-$\mathcal{S}'$ is considered as a prerequisite, which implies that the original closed-loop system (1)-(2) can be interpreted as a primal-dual gradient algorithm to solve an unconstrained quadratic saddle point problem. In this section, we investigate Class-$\mathcal{S}'$ by studying what kind of systems belongs to it. Generally speaking, a system should be at least marginally stable [36] if it can be interpreted as a primal-dual gradient algorithm for an unconstrained quadratic saddle point problem, since system trajectories are bounded in this case (Lemma 1). Also, the equilibrium set of the system should be equivalent to the saddle point set of the resulting problem. To limit the scope of discussion, we only focus on continuous-time LTI systems that are in accordance with (1)-(2).

Consider an autonomous LTI system given by

$$\dot{x} = Ax + Cw \tag{11}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times p}$, and $w(t) \in \mathbb{R}^p$ is the exogenous input, e.g., disturbances. In general, any given LTI closed-loop system with either static feedback or dynamic feedback can be rearranged to fit (11).

First, let us study the following class of LTI systems which can be interpreted as gradient algorithms for solving an unconstrained convex quadratic optimization problem. Investigating these systems provides insights for developing conditions on reformulating dynamical systems as certain optimization algorithms, and helps to interpret the results for Class-$\mathcal{S}'$.

Class-$\mathcal{O}$:[5] System (11) belongs to Class-$\mathcal{O}$ if there exists a function $L(x) : \mathbb{R}^n \to \mathbb{R}$ and a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that $\nabla_x^2 L \preceq 0$, $\{x | \nabla_x L = \mathbf{0}\}$ is nonempty, and (11) is a gradient algorithm to solve $\max_x L$, i.e., $\dot{x} = P \frac{\partial L}{\partial x}$.

Since (11) is linear, if (11) belongs to Class-$\mathcal{O}$, then the associated L must be a concave quadratic function of x, i.e., $L = \frac{1}{2} x^T Q x + x^T R(Cw) + S(w)$ for some matrices $Q \preceq 0$

[5]Notation $\mathcal{O}$ stands for *optimization algorithms* in contrast to $\mathcal{S}$ for *saddle-point algorithms*.

and R, and $S(w)$ stands for terms only related to w. Therefore, system (11) belongs to Class-$\mathcal{O}$ if and only if there exist $P \succ 0$ and $Q \preceq 0$ such that $A = PQ$ holds (also let $R = P^{-1}$ to make $\dot{x} = P\frac{\partial L}{\partial x}$ hold). This leads to the following result.

Theorem 2. *Let w be constant in system (11) and the set $\{x|Ax+Cw = \mathbf{0}\}$ be nonempty. System (11) belongs to Class-$\mathcal{O}$ if and only if (11) is marginally or asymptotically stable, $eig(A) \in \mathbb{R}$, and A is diagonalizable*[6].

Proof. Necessity. Suppose that system (11) is a gradient algorithm to solve an unconstrained convex quadratic programming problem given by

$$\max_{x\in\mathbb{R}^n} L = \frac{1}{2}x^TQx + x^TR(Cw) + S(w) \tag{12}$$

where $Q \in \mathbb{R}^{n\times n}$ satisfies $Q = Q^T \preceq 0$, $R \in \mathbb{R}^{n\times n}$, and $S(w)$ stands for terms consisting of only w. Then the trajectories of (11) are bounded under Lemma 1, and there exists a matrix $P = P^T \in \mathbb{R}^{n\times n}$ and $P \succ 0$, so that $\dot{x} = P\frac{\partial L}{\partial x} = PQx + PR(Cw) = Ax + Cw$ is always true, i.e., $A = PQ$ and $PR = I_n$ hold. This leads to $R = P^{-1}$ as well as $P^{-1}A = A^TP^{-1}$ which is equivalent to $\text{eig}(A) \in \mathbb{R}$ and A is diagonalizable, based on Lemma 4 in the Appendix. Also, (11) is marginally or asymptotically stable.

Sufficiency. Because system (11) is marginally or asymptotically stable, $\text{eig}(A) \in \mathbb{R}$ and A is diagonalizable, A can be written as a diagonal canonical form $A = J\Lambda J^{-1}$ where $J \in \mathbb{R}^{n\times n}$ and $\Lambda \preceq 0$. Under Lemma 4, there exists a positive definite matrix $V \in \mathbb{R}^{n\times n}$ so that $V\Lambda = \Lambda V$ holds. Based on Lemma 1 in [37], we have $V\Lambda \preceq 0$. Define a matrix $P = JV^{-1}J^T$. Then $P^{-1}A = A^TP^{-1} \preceq 0$ holds. Considering the unconstrained convex quadratic programming problem (12) where $Q = P^{-1}A$ and $R = P^{-1}$, we conclude that system (11) belongs to Class-$\mathcal{O}$. □

Theorem 2 proposes a necessary and sufficient condition to interpret (11) as a gradient algorithm of an unconstrained convex quadratic programming problem. We now study the case where we can interpret (11) as a primal-dual gradient algorithm to solve an unconstrained quadratic saddle point problem. Consider the following class of systems.

Class-$\mathcal{S}$: System (11) belongs to Class-$\mathcal{S}$ if there exists a function $L(x^{(1)}, x^{(2)}) : \mathbb{R}^n \to \mathbb{R}$ and positive definite matrices $P_{x^{(1)}}$, $P_{x^{(2)}}$ such that $\nabla^2_{x^{(1)}}L \preceq 0$, $\nabla^2_{x^{(2)}}L \succeq 0$, the saddle point set $\{x|\nabla_x L = \mathbf{0}\}$ is nonempty, and (11) is a primal-dual gradient algorithm to solve $\max_{x^{(1)}}\min_{x^{(2)}} L$, i.e., $\dot{x} = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}\}\frac{\partial L}{\partial x}$.

In the above definition, state x is partitioned into $x^{(1)}$ and $x^{(2)}$ (the partition may not be unique). Similarly, (11) is rearranged in the following form (in this section, we abuse notation e.g., $A_{ij}, n_i, \lambda_i, B_i, C_i$, whose meaning should be clear from the context)

$$\underbrace{\begin{bmatrix} \dot{x}^{(1)} \\ \dot{x}^{(2)} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} x + Cw \tag{13}$$

[6]In this paper, we use $\text{eig}(A) \in \mathbb{R}$ to indicate that all eigenvalues of A are real numbers.

where $x^{(1)}(t) \in \mathbb{R}^{n_1}$, $x^{(2)}(t) \in \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$. Using similar arguments as before, we only need to focus on a function L in quadratic form, i.e.,

$$L = \frac{1}{2}x^T \underbrace{\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}}_{Q} x + x^TR(Cw) + S(w) \tag{14}$$

where $Q_{11} \in \mathbb{R}^{n_1\times n_1}$ satisfies $Q_{11} = Q_{11}^T \preceq 0$ (i.e., L is concave in $x^{(1)}$), $Q_{22} \in \mathbb{R}^{n_2\times n_2}$ satisfies $Q_{22} = Q_{22}^T \succeq 0$ (i.e., L is convex in $x^{(2)}$), and $Q_{12} \in \mathbb{R}^{n_1\times n_2}$. Based on the definition of Class-$\mathcal{S}$, we have the following theorem.

Theorem 3. *Let w be constant in (13) and the set $\{x|Ax + Cw = \mathbf{0}\}$ be nonempty. System (13) belongs to Class-$\mathcal{S}$ if and only if the following conditions are satisfied: (i) (13) is marginally or asymptotically stable; (ii) the eigenvalues of both A_{11} and A_{22} are non-positive real; (iii) both A_{11} and A_{22} are diagonalizable with the diagonal canonical forms given by $A_{11} = J_1\Lambda_1J_1^{-1}, J_1 \in \mathbb{R}^{n_1\times n_1}, A_{22} = J_2\Lambda_2J_2^{-1}, J_2 \in \mathbb{R}^{n_2\times n_2}$, and there exist V_1 and V_2 such that*

$$\begin{aligned} &(J_1^{-1})^TV_1J_1^{-1}A_{12} + A_{21}^T(J_2^{-1})^TV_2J_2^{-1} = \mathbf{0} \\ &V_1\Lambda_1 = \Lambda_1V_1,\ V_2\Lambda_2 = \Lambda_2V_2 \\ &V_1 \succ 0,\ V_2 \succ 0. \end{aligned} \tag{15}$$

Proof. Necessity. Suppose that (13) is a primal-dual gradient algorithm to solve $\max_{x^{(1)}}\min_{x^{(2)}} L$ where L is given by (14). Then the trajectories of (13) are bounded under Lemma 1, and there exist matrices $P_{x^{(1)}} \in \mathbb{R}^{n_1\times n_1}$ and $P_{x^{(2)}} \in \mathbb{R}^{n_2\times n_2}$ satisfying $P_{x^{(1)}} = P_{x^{(1)}}^T \succ 0$ and $P_{x^{(2)}} = P_{x^{(2)}}^T \succ 0$, so that $R = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}\}^{-1}$ and $Q = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}\}^{-1}A = A^T\text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}\}^{-1}$ holds. The last equation leads to

$$P_{x^{(1)}}^{-1}A_{11} = A_{11}^TP_{x^{(1)}}^{-1} = Q_{11} \preceq 0 \tag{16a}$$

$$P_{x^{(2)}}^{-1}A_{22} = A_{22}^TP_{x^{(2)}}^{-1} = -Q_{22} \preceq 0 \tag{16b}$$

$$P_{x^{(1)}}^{-1}A_{12} + A_{21}^TP_{x^{(2)}}^{-1} = \mathbf{0}. \tag{16c}$$

Based on Lemma 4 in the Appendix, (16a)-(16b) are equivalent to $\text{eig}(A_{11}) \in \mathbb{R}$, A_{11} is diagonalizable, $\text{eig}(A_{22}) \in \mathbb{R}$, A_{22} is diagonalizable, and moreover, the eigenvalues of both A_{11} and A_{22} are non-positive (otherwise, for example, there would exist a positive eigenvalue of A_{11} and a corresponding non-zero eigenvector, denoted by λ_1 and x_{λ_1}, for which $x_{\lambda_1}^TP_{x^{(1)}}^{-1}A_{11}x_{\lambda_1} = \lambda_1x_{\lambda_1}^TP_{x^{(1)}}^{-1}x_{\lambda_1} > 0$, i.e., a contradiction to (16a)). By defining $V_1 = J_1^TP_{x^{(1)}}^{-1}J_1$, $V_2 = J_2^TP_{x^{(2)}}^{-1}J_2$, condition (iii) holds, which completes the proof of necessity.

Sufficiency. Let conditions (i)-(iii) be true. Consider the following unconstrained quadratic saddle point problem:

$$\max_{x^{(1)}\in\mathbb{R}^{n_1}}\ \min_{x^{(2)}\in\mathbb{R}^{n_2}} L = \frac{1}{2}x^TP^{-1}Ax + x^TP^{-1}(Cw) + S(w) \tag{17}$$

where $P^{-1} = \text{diag}\{(J_1^{-1})^TV_1J_1^{-1}, -(J_2^{-1})^TV_2J_2^{-1}\}$. Due to $V_1\Lambda_1 \preceq 0, V_2\Lambda_2 \preceq 0$, L is concave in $x^{(1)}$ and convex in $x^{(2)}$. Define matrices $P_{x^{(1)}} = J_1V_1^{-1}J_1^T \succ 0$, $P_{x^{(2)}} = J_2V_2^{-1}J_2^T \succ 0$. Under Lemma 1, the trajectories of the primal-dual gradient algorithm $\dot{x} = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}\}\frac{\partial L}{\partial x}$ are bounded, which is the same as (13). So we conclude that (13) is in Class-$\mathcal{S}$. □

Theorem 3 establishes a necessary and sufficient condition to interpret (13) as a primal-dual gradient algorithm of an unconstrained quadratic saddle point problem. One of the conditions is to check the feasibility of (15). This problem is actually a Semi-Definite Programming (SDP) problem which can be solved using SDP solvers, e.g., SeDuMi [38]. Similar to Remark 4, adding a minus sign before L leads to an alternative formulation of (17) in which $x^{(1)}$ appears as a minimizer and $x^{(2)}$ appears as a maximizer.

Because Class-$\mathcal{S}'$ is a subset of Class-$\mathcal{S}$, we can apply the above results to system (1)-(2) and Class-$\mathcal{S}'$. Partition (1) in the form of (13) and rearrange (1)-(2) as

$$\underbrace{\begin{bmatrix} \dot{x}^{(1)} \\ \dot{x}^{(2)} \\ \dot{u} \end{bmatrix}}_{\dot{\tilde{x}}} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ D_1 & D_2 & E \end{bmatrix}}_{\tilde{A}} \tilde{x} + \underbrace{\begin{bmatrix} C_1 \\ C_2 \\ F \end{bmatrix}}_{\tilde{C}} w \qquad (18)$$

where $x^{(1)}(t) \in \mathbb{R}^{n_1}$, $x^{(2)}(t) \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$. Then the following corollary is immediate.

Corollary 1. *Let w be constant in (18) and the set $\{\tilde{x} | \tilde{A}\tilde{x} + \tilde{C}w = \mathbf{0}\}$ be nonempty. System (18) belongs to Class-$\mathcal{S}'$ if and only if the following conditions are satisfied: (i) (18) is marginally or asymptotically stable; (ii) the eigenvalues of A_{11}, A_{22}, E, $E_{ii}, i = 1, \cdots, N$ and $\begin{bmatrix} A_{22} & B_2 \\ D_2 & E \end{bmatrix}$ are non-positive real, and these matrices are diagonalizable; (iii) let the diagonal canonical forms of A_{11}, A_{22}, $E_{ii}, i = 1, \cdots, N$ be $A_{11} = J_1 \Lambda_1 J_1^{-1}, J_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_{22} = J_2 \Lambda_2 J_2^{-1}, J_2 \in \mathbb{R}^{n_2 \times n_2}$, $E_{ii} = J_{E_i} \Lambda_{E_i} J_{E_i}^{-1}, J_{E_i} \in \mathbb{R}^{m_i \times m_i}$, there exist V_1, V_2, V_{E_i} such that*

$$(J_1^{-1})^T V_1 J_1^{-1} A_{12} + A_{21}^T (J_2^{-1})^T V_2 J_2^{-1} = \mathbf{0}$$
$$(J_1^{-1})^T V_1 J_1^{-1} B_1 + D_1^T \mathit{diag}\{(J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\} = \mathbf{0}$$
$$(J_2^{-1})^T V_2 J_2^{-1} B_2 - D_2^T \mathit{diag}\{(J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\} = \mathbf{0}$$
$$(J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1} E_{ij} - E_{ji}^T (J_{E_j}^{-1})^T V_{E_j} J_{E_j}^{-1} = \mathbf{0}, \ j \in \mathcal{N}(i)$$
$$V_1 \Lambda_1 = \Lambda_1 V_1, \ V_2 \Lambda_2 = \Lambda_2 V_2, \ V_{E_i} \Lambda_{E_i} = \Lambda_{E_i} V_{E_i} \ \textit{and}$$
$$V_1 \succ 0, \ V_2 \succ 0, \ V_{E_i} \succ 0 \ \textit{hold, where } i = 1, \cdots, N.$$

Proof. See the Appendix. □

V. A Practical Example

In this section, we provide an example on frequency control in power networks to demonstrate the use of our control modification procedure. In [20], a decentralized integral control scheme is investigated to restore system frequency in the presence of disturbances. Regarding this scheme as a built-in control mechanism, we apply the proposed procedure to redesign the closed-loop system that belongs to Class-$\mathcal{S}'$. The objective is not only to restore the frequency, but also to result in a better supply-demand balance respecting system operating constraints, which can be described as a DC OPF problem given in [28] (Problem (5.2) in Page 100). The exact form of the redesigned closed-loop system can be found in [28] (Equation (5.11) in Page 105).

We use an IEEE 14-bus power system illustrated in Figure 1 as an example. The parameters of the network are provided

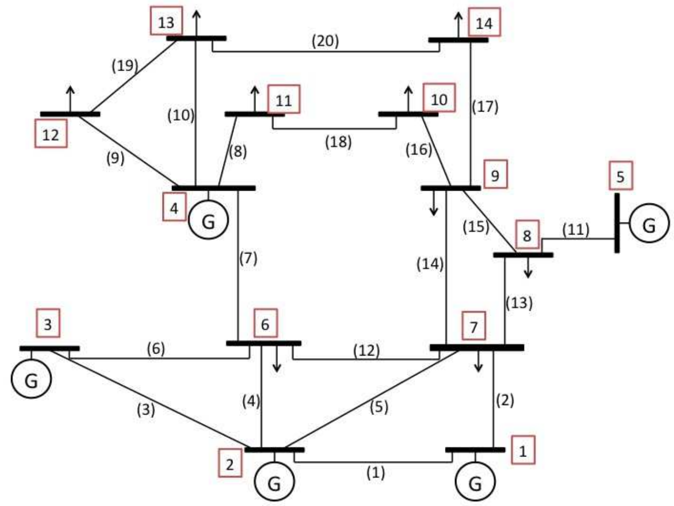

Fig. 1: IEEE 14-bus network, with 5 generator buses, 9 load buses and 20 transmission lines. The buses are numbered with red blocks. The transmission lines are numbered with brackets.

in Table 5.2 in [28]. We consider a scenario consisting of two disturbances which is realized as follows: the system is stabilized at the nominal operating point at $t = 0$s; at $t = 5$s, there is a step change $+2.5$p.u. (i.e., per unit) at bus 6; 40s later, there is a step change -3p.u. at bus 9. The simulation results are shown in Figures 2 and 4, where we also compare with the integral controller from [20]. It is obvious that the modified controller leads to less oscillations and less cost, and moreover, satisfies operating constraints (in the case of using integral control, load power consumption at buses 6, 7, 9, 13, 14 does not satisfy the constraints, and the power flow in lines 7, 13, 14 does not satisfy the constraints). If we change the disturbances to be time-varying, the simulation results are shown in Figures 3-4. In this case, the modified controller still leads to a lower cost and satisfies operating constraints (when using integral control, load power consumption at buses 6, 14 does not satisfy the constraints).

To conclude, the redesigned control can be added to the current primary and secondary control for disturbance rejection and frequency restoration in power systems, as well as achieving a higher economic efficiency because the redesigned controller can drive the system to a state that is optimal to some given economic dispatch problem. From the simulation results, we can see that generators and loads share the disturbance more evenly, corresponding to a lower aggregated operating cost. We would also like to remark that here the frequency deviation is around 0.05Hz. Though reliability requirement may be 0.5Hz, in many real systems such as ERCOT, the threshold for frequency control is around 0.05Hz. The simulation results show that the controller is able to maintain the frequency within this range.

VI. Conclusion and Future Work

In this paper, we have studied distributed control for a class of linear network systems to achieve optimal steady-state performance via a control retrofit framework. This framework

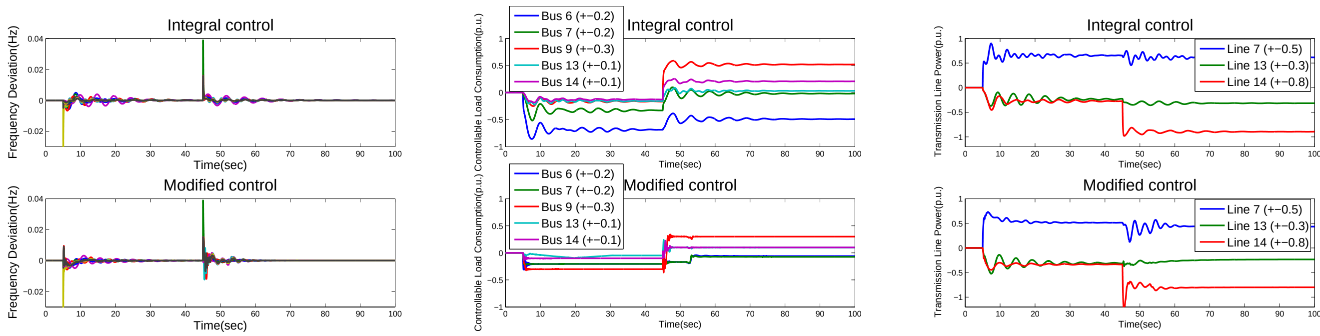

Fig. 2: Simulation under constant disturbances. Left: bus frequency (deviations). Middle: controllable load power consumption (deviations) at selected buses. Right: power flow (deviations) in selected lines. Capacities are indicated in the legend.

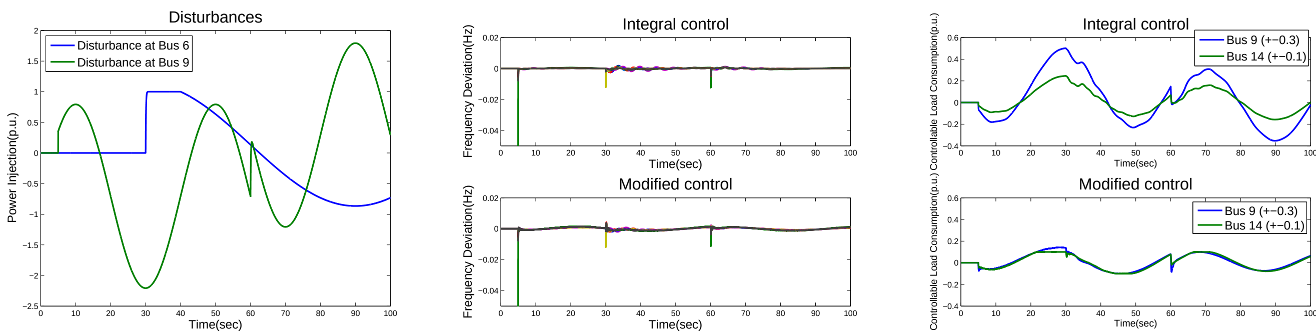

Fig. 3: Simulation under time-varying disturbances. Left: disturbance injections. Middle: bus frequency (deviations). Right: controllable load power consumption (deviations) at selected buses. Capacities are indicated in the legend.

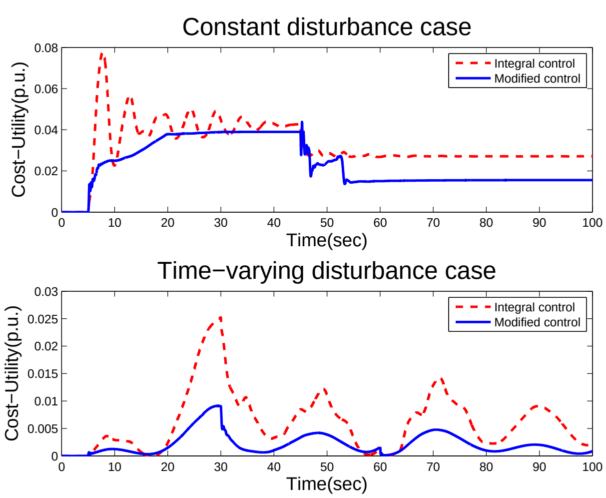

Fig. 4: Operating cost comparison.

consists of two steps: seek an appropriate optimization problem that the system dynamics implicitly solve; then modify the resulting optimization problem by incorporating a predefined optimization problem and derive control mechanisms to solve the augmented problem. In order to investigate how general this framework is, we have developed necessary and sufficient conditions under which an LTI system can be interpreted as a gradient algorithm of either an unconstrained convex optimization problem or an unconstrained saddle point problem.

In the future, we will investigate the performance of the overall system under time-varying disturbances (since the dynamics of the overall system are primal-dual gradient dynamics of a saddle point problem, literature on the convergence rate of primal-dual gradient algorithms could assist the analysis). We will also extend our result to discrete-time LTI systems, and then focus on (linear) time-varying systems. Another research direction is to improve the retrofit procedure when there is only limited access (measurement) to system states and parameters. Last but not least, we will apply our result to other networked dynamical systems such as mechanical systems, robot network systems and thermal dynamics in buildings.

REFERENCES

[1] X. Zhang, A. Papachristodoulou, and N. Li, "Distributed optimal steady-state control using reverse- and forward-engineering," in *Proc. of 54th IEEE Conference on Decision and Control*, 2015, pp. 5257–5264.

[2] H. Farhangi, "The path of the smart grid," *IEEE Power Energy Magazine*, vol. 8, no. 1, pp. 18–28, 2010.

[3] W. Shi, N. Li, X. Xie, C.-C. Chu, and R. Gadh, "Optimal residential demand response in distribution network," *IEEE Journal on Selected Areas in Communications*, vol. 32, no. 7, pp. 1441–1450, 2014.

[4] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[5] F.-Y. Wang, "Parallel control and management for intelligent transportation systems: Concepts, architectures, and applications," *IEEE Transactions on Intelligent Transportation Systems*, vol. 11, no. 3, pp. 630–638, 2010.

[6] N. Li, L. Chen, C. Zhao, and S. H. Low, "Connecting automatic generation control and economic dispatch from an optimization view," in *Proc. of 2014 American Control Conference*, 2014, pp. 735–740.

[7] D. D. Šiljak, *Large-Scale Dynamic Systems: Stability and Structure*. North Holland, New York, 1978.

[8] J. L. Speyer, "Computation and transmission requirements for a decentralized linear-quadratic-Gaussian control problem," *IEEE Transactions on Automatic Control*, vol. 24, no. 2, pp. 266–269, 1979.

[9] D. D. Šiljak, *Decentralized Control of Complex Systems*. Academic Press, Boston, 1991.

[10] R. D'Andrea and G. E. Dullerud, "Distributed control design for spatially interconnected systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1478–1495, 2003.

[11] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 274–286, 2006.

[12] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. of IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[13] A. Rantzer, "Distributed control of positive systems," *arXiv:1203.0047v3 [math.OC]*, 2012.

[14] C. H. Papadimitriou and J. Tsitsiklis, "Intractable problems in control theory," *SIAM Journal on Control and Optimization*, vol. 24, no. 4, pp. 639–654, 1986.

[15] C. Zhao, U. Topcu, N. Li, and S. Low, "Design and stability of load-side primary frequency control in power systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 5, pp. 1177–1189, 2014.

[16] S. You and L. Chen, "Reverse and forward engineering of frequency control in power networks," in *Proc. of 53rd IEEE Conference on Decision and Control*, 2014, pp. 191–198.

[17] X. Zhang, N. Li, and A. Papachristodoulou, "Achieving real-time economic dispatch in power networks via a saddle point design approach," in *Proc. of IEEE PES Power and Energy Society General Meeting*, 2015.

[18] M. Chiang, S. H. Low, A. R. Calderbank, and J. C. Doyle, "Layering as optimization decomposition: A mathematical theory of network architectures," *Proc. of IEEE*, vol. 95, no. 1, pp. 255–312, 2007.

[19] J. Wang and N. Elia, "A control perspective for centralized and distributed convex optimization," in *Proc. of 50th IEEE Conference on Decision and Control*, 2011, pp. 3800—3805.

[20] C. Zhao, E. Mallada, and F. Dörfler, "Distributed frequency control for stability and economic dispatch in power systems," in *Proc. of 2015 American Control Conference*, 2015, pp. 2359–2364.

[21] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[22] P. Kundur, *Power System Stability and Control*. New York: McGraw-Hill, 1994.

[23] M. Andreasson, D. V. Dimarogonas, H. Sandberg, and K. H. Johansson, "Distributed control of networked dynamical systems: Static feedback, integral action and consensus," *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1750–1764, 2014.

[24] A. Jokic, "Price-based optimal control of electrical power systems," *PhD dissertation, Department of Electrical Engineering, Eindhoven University of Technology*, 2007.

[25] F. D. Brunner, H.-B. Dürr, and C. Ebenbauer, "Feedback design for multi-agent systems: A saddle point approach," in *Proc. of 51st IEEE Conference on Decision and Control*, 2012, pp. 3783–3789.

[26] E. Dall'Anese, S. Dhople, and G. B. Giannakis, "Regulation of dynamical systems to optimal solutions of semidefinite programs: Algorithms and applications to AC optimal power flow," in *Proc. of 2015 American Control Conference*, 2015, pp. 2087–2092.

[27] D. Feijer and F. Paganini, "Stability of primal-dual gradient dynamics and applications to network optimization," *Automatica*, vol. 46, no. 12, pp. 1974–1981, 2010.

[28] X. Zhang, "Scalable (re)design frameworks for optimal, distributed control in power networks," *PhD dissertation, University of Oxford*, 2015.

[29] X. Zhang and A. Papachristodoulou, "A distributed PID controller for network congestion control problems," in *Proc. of 2014 American Control Conference*, 2014, pp. 5453–5458.

[30] ——, "Improving the performance of network congestion control algorithms," *IEEE Transactions on Automatic Control*, vol. 60, no. 2, pp. 522–527, 2015.

[31] D. P. Bertsekas, *Constrained Optimization and Lagrange Multiplier Methods*. Athena Scientific, 1996.

[32] ——, *Nonlinear Programming*. Athena Scientific, Belmont, Massachusetts, 1999.

[33] J. Nocedal and S. J. Wright, *Numerical Optimization*. Springer-Verlag New York, Inc, 1999.

[34] D. P. Bertsekas, A. Nedich, and A. Ozdaglar, *Convex analysis and optimization*. Athena Scientific, Belmont, Massachusetts, 2003.

[35] R. Srikant, *The Mathematics of Internet Congestion Control*. Cambridge, MA: Birkhäuser, 2004.

[36] C.-T. Chen, *Linear System Theory and Design*, 3rd ed. Oxford University Press, Inc. New York, 1999.

[37] G. Vinnicombe, "On the stability of end-to-end congestion control for the internet," in *Technical Report, University of Cambridge, CUED/F-INFENG/TR.398*, 2000.

[38] J. F. Sturm, "Using SEDUMI 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11, no. 12, pp. 625–653, 1999.

[39] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, Upper Saddle River, New Jersey, 2002.

Appendix

Proof Sketch of Theorem 1: Since A has no eigenvalues on the imaginary axis, therefore invertible, under Lemma 2, at any equilibrium point of (8), (y, u) is an optimal solution of problem (3). The proof of convergence is similar to that in Lemma 3 (i.e., Lemma 5.4 in [28] (pp. 113–114)), by constructing a quadratic Lyapunov function and showing that its derivative with respect to time is non-increasing along the trajectory of system (8). When this derivative is 0, $u_i = \hat{u}_i$, $y_i = \hat{y}_i$, $\zeta_i = \hat{\zeta}_i$, $\lambda_i = \hat{\lambda}_i$ hold, which leads to $\dot{u}_i = \mathbf{0}, \dot{y}_i = \mathbf{0}, \dot{\zeta}_i = \mathbf{0}, \dot{\lambda}_i = \mathbf{0}$. Given constant u and w, system (1) eventually converges to an equilibrium point at which $x = y$ holds, because A is Hurwitz (since under constant u and w, system (1) can also be interpreted as a primal-dual algorithm for solving a quadratic saddle point problem based on Corollary 1, i.e., system (1) is marginally or asymptotically stable; moreover, A has no eigenvalues on the imaginary axis). Then $\dot{\mu}_i = 0, i = 1, \cdots, N$ are true (otherwise, there exists i such that $h_i(y_i, u_i, y_j, u_j) > 0$ and $\nabla_{y_i,u_i,y_j,u_j} h_i = \mathbf{0}$ hold, which lead to that h_i is always positive due to its convexity, i.e., a contradiction to the feasibility of the optimization problem). From LaSalle's invariance principle [39], each trajectory of (8) asymptotically converges to an equilibrium point at which (x, u) is an optimal solution of problem (3). □

Lemma 4. *Given $A \in \mathbb{R}^{n\times n}$, the following statements are equivalent:*

(i) There exists a matrix $P = P^T \in \mathbb{R}^{n\times n}$ and $P \succ 0$, so that $PA = A^T P$ holds;

(ii) eig$(A) \in \mathbb{R}$ and A is diagonalizable.

Proof. (i)⇒(ii). Write A in its Jordan canonical form $A = J\Lambda J^{-1}$, where $\Lambda = \text{diag}\{\Lambda_1, \cdots, \Lambda_a\}$ and each Λ_i is a Jordan block. Without loss of generality, let

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & 1 & \mathbf{0} \\ 0 & \lambda_1 & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots \end{bmatrix}$$

and the dimension of Λ_1 is at least 2×2. Then (i) leads to $PJ\Lambda J^{-1} = (J^{-1})^H \Lambda^H J^H P$ which is equivalent to

$$J^H PJ\Lambda = \Lambda^H J^H PJ. \tag{19}$$

Note that $J^H PJ$ is a Hermitian matrix, and $J^H PJ \succ 0$ since $P \succ 0$ and J is invertible. So we obtain that all the diagonal entries of $J^H PJ$ are positive real. Let

$$J^H PJ = \begin{bmatrix} s_{11} & s_{12} & \cdots \\ s_{12}^H & s_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

where $s_{11}, s_{22} > 0$. From (19), we have $\lambda_1 s_{11} = \lambda_1^H s_{11}$ and $s_{11} + s_{12}\lambda_1 = \lambda_1^H s_{12}$ which result in $\lambda_1 = \lambda_1^H$ and $s_{11} = 0$, i.e., a contradiction. Therefore, each Jordan block should have the dimension 1×1 and a real diagonal entry, i.e., A is diagonalizable and $\text{eig}(A) \in \mathbb{R}$ are true.

(ii)⇒(i). Based on (ii), write A as its diagonal canonical form $A = J\Lambda J^{-1}$, where Λ is a diagonal matrix with real entries on the diagonal and $J \in \mathbb{R}^{n\times n}$. Define a matrix $V \in \mathbb{R}^{n\times n}$ so that $V = V^T \succ 0$ and $V\Lambda = \Lambda V$ hold. Such a matrix V always exists and the simplest choice is $V = I_n$. We can then find a matrix $P = P^T = (J^{-1})^T V J^{-1} \in \mathbb{R}^{n\times n}$ so that $P \succ 0$ and $PA = A^T P$ hold, i.e., (i) is true. □

Proof of Corollary 1: Necessity. Suppose that system (18) belongs to Class-$\mathcal{S}'$ and is a primal-dual gradient algorithm to solve $\max_{x^{(1)}} \min_{x^{(2)},u} L_{\text{sys}}$,

$$L_{\text{sys}} = \frac{1}{2}\tilde{x}^T \underbrace{\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}}_{Q} \tilde{x} + \tilde{x}^T P^{-1}\tilde{C}w$$

where $Q_{11} \in \mathbb{R}^{n_1\times n_1}$ satisfies $Q_{11} = Q_{11}^T \preceq 0$ (i.e., L_{sys} is concave in $x^{(1)}$), $Q_{22} \in \mathbb{R}^{(n_2+m)\times(n_2+m)}$ satisfies $Q_{22} = Q_{22}^T \succeq 0$ (i.e., L_{sys} is convex in $(x^{(2)}, u)$), $Q_{12} \in \mathbb{R}^{n_1\times(n_2+m)}$ and $P \in \mathbb{R}^{(n+m)\times(n+m)}$. Then the trajectories of (18) are bounded under Lemma 1, and there exist matrices $P_{x^{(1)}} \in \mathbb{R}^{n_1\times n_1}$, $P_{x^{(2)}} \in \mathbb{R}^{n_2\times n_2}$ and $P_{u_i} \in \mathbb{R}^{m_i\times m_i}$ satisfying $P_{x^{(1)}} = P_{x^{(1)}}^T \succ 0$, $P_{x^{(2)}} = P_{x^{(2)}}^T \succ 0$ and $P_{u_i} = P_{u_i}^T \succ 0, i = 1, \cdots, N$, so that $Q = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}, -\text{diag}\{P_{u_i}\}\}^{-1}\tilde{A} = \tilde{A}^T \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}, -\text{diag}\{P_{u_i}\}\}^{-1}$ holds (also let $P = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}, -\text{diag}\{P_{u_i}\}\}$). This leads to

$$P_{x^{(1)}}^{-1} A_{11} = A_{11}^T P_{x^{(1)}}^{-1} = Q_{11} \preceq 0$$
$$\begin{bmatrix} P_{x^{(2)}}^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{P_{u_i}^{-1}\} \end{bmatrix} \begin{bmatrix} A_{22} & B_2 \\ D_2 & E \end{bmatrix} =$$
$$\begin{bmatrix} A_{22}^T & D_2^T \\ B_2^T & E^T \end{bmatrix} \begin{bmatrix} P_{x^{(2)}}^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{P_{u_i}^{-1}\} \end{bmatrix} = -Q_{22} \preceq 0$$
$$P_{x^{(1)}}^{-1} \begin{bmatrix} A_{12} & B_1 \end{bmatrix} =$$
$$-\begin{bmatrix} A_{21}^T & D_1^T \end{bmatrix} \begin{bmatrix} P_{x^{(2)}}^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{P_{u_i}^{-1}\} \end{bmatrix}$$

which can be further rearranged as

$$P_{x^{(1)}}^{-1} A_{11} = A_{11}^T P_{x^{(1)}}^{-1} \preceq 0$$
$$P_{x^{(2)}}^{-1} A_{22} = A_{22}^T P_{x^{(2)}}^{-1} \preceq 0$$
$$P_{x^{(2)}}^{-1} B_2 = D_2^T \text{diag}\{P_{u_i}^{-1}\}$$
$$\text{diag}\{P_{u_i}^{-1}\} \begin{bmatrix} E_{11} & \cdots & E_{1N} \\ \cdots & \cdots & \cdots \\ E_{N1} & \cdots & E_{NN} \end{bmatrix} =$$
$$\begin{bmatrix} E_{11}^T & \cdots & E_{N1}^T \\ \cdots & \cdots & \cdots \\ E_{1N}^T & \cdots & E_{NN}^T \end{bmatrix} \text{diag}\{P_{u_i}^{-1}\} \preceq 0$$
$$P_{x^{(1)}}^{-1} A_{12} + A_{21}^T P_{x^{(2)}}^{-1} = \mathbf{0}$$
$$P_{x^{(1)}}^{-1} B_1 + D_1^T \text{diag}\{P_{u_i}^{-1}\} = \mathbf{0}.$$

Based on Lemma 4, the above equations are equivalent to conditions (ii) and (iii) by defining $V_1 = J_1^T P_{x^{(1)}}^{-1} J_1, V_2 = J_2^T P_{x^{(2)}}^{-1} J_2, V_{E_i} = J_{E_i}^T P_{u_i}^{-1} J_{E_i}, i = 1, \cdots, N$.

Sufficiency. Let conditions (i)-(iii) be true. Consider the following unconstrained quadratic saddle point problem:

$$\max_{x^{(1)}\in\mathbb{R}^{n_1}} \min_{x^{(2)}\in\mathbb{R}^{n_2}, u\in\mathbb{R}^m} L_{\text{sys}} = \frac{1}{2}\tilde{x}^T P^{-1}\tilde{A}\tilde{x} + \tilde{x}^T P^{-1}\tilde{C}w$$

where $P^{-1} = \text{diag}\{(J_1^{-1})^T V_1 J_1^{-1}, -(J_2^{-1})^T V_2 J_2^{-1}, -\text{diag}\{(J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\}\}$. Due to $V_1\Lambda_1 \preceq 0$ and

$$\begin{bmatrix} (J_2^{-1})^T V_2 J_2^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\} \end{bmatrix} \begin{bmatrix} A_{22} & B_2 \\ D_2 & E \end{bmatrix} =$$
$$\begin{bmatrix} A_{22}^T & D_2^T \\ B_2^T & E^T \end{bmatrix} \begin{bmatrix} (J_2^{-1})^T V_2 J_2^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\} \end{bmatrix}$$
$$\preceq 0 \tag{20}$$

L_{sys} is concave in $x^{(1)}$ and convex in $(x^{(2)}, u)$. Equation (20) is derived as follow. Let $\tilde{A}_{22} = \begin{bmatrix} A_{22} & B_2 \\ D_2 & E \end{bmatrix}$ and $\tilde{P}_2^{-1} = \begin{bmatrix} (J_2^{-1})^T V_2 J_2^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\} \end{bmatrix}$. Since the eigenvalues of $\tilde{A}_{22}$ are non-positive real and $\tilde{A}_{22}$ is diagonalizable, $\tilde{A}_{22}$ can be written as $\tilde{A}_{22} = \tilde{J}\tilde{\Lambda}\tilde{J}^{-1}$ where $\tilde{\Lambda} \preceq 0$ is a diagonal matrix. Due to $\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda}\tilde{J}^{-1} = (\tilde{J}^{-1})^T\tilde{\Lambda}\tilde{J}^T\tilde{P}_2^{-1}$, based on Lemma 1 in [37], $\tilde{J}^T\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda} = \tilde{\Lambda}\tilde{J}^T\tilde{P}_2^{-1}\tilde{J} \preceq 0$ holds. For any given vector $v \in \mathbb{R}^{n_2+m}$, $(\tilde{J}v)^T\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda}\tilde{J}^{-1}(\tilde{J}v) = v^T\tilde{J}^T\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda}v \leq 0$, which leads to $\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda}\tilde{J}^{-1} \preceq 0$, i.e., Equation (20) holds.

Now define matrices $P_{x^{(1)}} = J_1 V_1^{-1} J_1^T \succ 0$, $P_{x^{(2)}} = J_2 V_2^{-1} J_2^T \succ 0$ and $P_{u_i} = J_{E_i} V_{E_i}^{-1} J_{E_i}^T \succ 0, i = 1, \cdots, N$. Under Lemma 1, the trajectories of the primal-dual gradient algorithm given by $\dot{x} = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}\}\frac{\partial L_{\text{sys}}}{\partial x}$ and $\dot{u}_i = -P_{u_i}\frac{\partial L_{\text{sys}}}{\partial u_i}, i = 1, \cdots, N$ are bounded, which is the same as (18). So we conclude that (18) is in Class-$\mathcal{S}'$. □