On the Relationship between the VCG Mechanism and Market Clearing

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***Abstract*— We consider a social cost minimization problem with equality and inequality constraints in which a central coordinator allocates infinitely divisible goods to self-interested N firms under information asymmetry. We consider the Vickrey-Clarke-Groves (VCG) mechanism and study its connection to an alternative mechanism based on market clearing-price. Under the considered set up, we show that the VCG payments are equal to the path integrals of the vector field of the market clearing prices, indicating a close relationship between the VCG mechanism and the "clearing-price" mechanism. We then discuss its implications for the electricity market design and also exploit this connection to analyze the budget balance of the VCG mechanism.**

I. Introduction

Market power monitoring and mitigation are essential aspects of today's electricity market operations [1]. The efficiency loss due to market manipulations by strategic firms are not only predicted in theory [2, Ch. 12], but also documented as real incidents [3]. While the current practice of market power monitoring relies on relatively simple metrics (e.g., Herfindahl–Hirschman Index [4]), further deregulation and emergence of new markets (e.g., ISO-aggregator and aggregator-prosumer interfaces) necessitate advanced market power analysis [5] and appropriate market design.

The efficiency loss in oligopolistic electricity markets has been actively studied in recent literature. Among others, a popular venue of research (e.g., [6]–[9]) is based on the supply function bidding model by Klemperer and Meyer [10], in which the market clears according to the producers' (resp. consumers') submitted supply (resp. demand) functions. In [10], it is observed that if there is no demand uncertainty and arbitrary supply function bidding is allowed, numerous multiplicity of equilibria emerges. Johari and Tsitsiklis [8] observed that "as the strategic flexibility granted to firms increase, their temptation to misdeclare their cost increases as well," and considered a simple (scalar-parametrized) supply function model with N participants and showed that the price of anarchy (efficiency loss) is upper bounded by $1 + 1/(N - 2)$. In [11], it is observed that the worst case efficiency loss depends on the transmission network topology when there exist transmission constraints. The efficiency loss under transmission constraints is further studied in [12] under a slightly different supply function bidding model.

While the market analysis via the supply function bidding model has been fruitful, there are several limitations as well. First, it may not be easy to guarantee the existence of supply function equilibria in realistic electricity markets. As demonstrated by [13], a pure Nash equilibrium may fail to exist even in a relatively simple supply function bidding model. Second, even if the supply function equilibria are known to exist, it could be a difficult task for the firms to find them efficiently [14]. Finally, unlike the mechanism design approach (discussed next), the efficiency loss is inevitable.

The *mechanism design* is an alternative approach towards the market power mitigation. For instance, the references [15]–[18] consider applications of the Vickrey-Clarke-Groves (VCG) mechanism [2, Ch. 23] to several market design problems in power system operations. The VCG mechanism provides an attractive market model since it attains zero efficiency loss and incentive compatibility (and often individual rationality [19]). It is also attractive since the existence of equilibria is guaranteed (truth-telling is an equilibrium) and it can be trivially found by the participants. However, a major shortcoming of the VCG mechanism is the lack of budget balance [8], which is a prominent reason why it is rarely used in practice. The VCG mechanism has other drawbacks such as vulnerability to coalitions, high computational complexity, and lack of privacy [19].

For a successful electricity market design in the future, an integrated discussion on these different approaches are desired in the common framework, so that their pros and cons are systematically understood. Therefore, in this paper, we discuss different market mechanisms, namely the "clearing-price" mechanism (the underlying mechanism for the supply function bidding model) and the VCG mechanism, in a common set up. As a common set up, we consider a general social cost minimization problem with equality and inequality constraints. Our goal is to provide a unified view on these seemingly different market mechanisms, which would hopefully be valuable in the future studies on electricity market design.

The contribution of this paper is two-fold. First, we show a connection between the VCG mechanism and the clearing-price mechanism by proving that the VCG payment is equal to the path integral of the vector fields of market clearing prices under the considered set up (Theorem 2). It also follows from this observation that tax values calculated by the clearing-price mechanism and the VCG mechanism are similar when individual participants' market power is negligible. This result indicates a close connection between the VCG and the clearing-price mechanisms. Second, in order to address the budget balance issue of the VCG mechanism, we apply the above observation to analyze the worst case budget loss in the VCG mechanism. With some simplified

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assumptions, we outline how the budget loss can be analyzed using the main result of this paper.

This paper is formulated as follows. In Section II, a general social cost minimization problem to be considered is formulated. The clearing-price mechanism and the VCG mechanism are described in Section III. The main technical result (path integral characterization of the VCG mechanism) is presented in Section IV. The budget loss of the VCG mechanism is studied in Section V.

II. Problem formulation

Suppose that there exist N participating firms indexed by $i = 1, 2, ..., N$ and a single auctioneer. Consider a social cost minimization problem with equality and inequality constraints:

$$\mathcal{P}: \quad \min_{x} \; f(x) \triangleq \sum\nolimits_{i=1}^{N} f_i(x_i) \tag{1a}$$

$$\text{s.t.} \;\; g(x) \triangleq \sum\nolimits_{i=1}^{N} g_i(x_i) \leq 0 \tag{1b}$$

$$h(x) \triangleq \sum\nolimits_{i=1}^{N} h_i(x_i) = 0. \tag{1c}$$

For each $i = 1, 2, ..., N$, x_i can be interpreted as production by the i-th firm, or as consumption if it is negative. The function $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$ is interpreted as a production cost if the firm i is a producer, or as a negated utility function if the firm is a consumer. The aggregation $f(x)$ is referred to as the social cost function (negated social welfare function). Functions $g_i : \mathbb{R}^{n_i} \to \mathbb{R}^k$ and $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^l$ are interpreted as the i-th firm's contribution to the global equality/inequality constraints such as the power flow balance constraints, line flow limit constraints, local capacity constraints. Constraints (1b) and (1c) are imposed entry-wise.

Remark 1: We remark here that in the problem setting (1), we only consider separable objective function and constraints function. This is usually true for electricity market, especially when DC power flow model is used where all the constraints are in a linear form [20].

We assume that functions g_i and h_i for each $i = 1, 2, ..., N$ are publicly known. However, the function $f_i \in \mathcal{F}_i$ is assumed to be private, i.e., known to the i-th firm only. The family $\mathcal{F}_i$ of functions are assumed to be known *a priori*. An optimal solution to $\mathcal{P}$ will be denoted by $x^* = (x_1^*; ...; x_N^*)$[1].

A. Bidding and payment model

Consider a single-round auction mechanism in which each firm is first given an opportunity to report his/her private function $\tilde{f}_i \in \mathcal{F}_i$ to the auctioneer. In this step, firms are allowed to misreport their information and thus $\tilde{f}_i \neq f_i$ in general. In the next step, the auctioneer makes a social decision $\tilde{x}^* = (\tilde{x}_1^*; ...; \tilde{x}_N^*)$ based on the reported information according to a certain *decision rule*. Finally, the auctioneer calculates payments $\pi = (\pi_1; ...; \pi_N) \in \mathbb{R}^N$ according to a certain *payment rule*. Throughout this paper, we consider π_i as the payment from the i-th firm to the auctioneer, which can be either positive or negative. Here π_i can be thought of as a tax by which the i-th firm's incentive is manipulated. The net cost for the i-th firm is $C_i = f_i(\tilde{x}_i^*) + \pi_i$. An auction mechanism is characterized by a particular choice of a decision rule and a payment rule.

We say that an auction mechanism is *(dominant strategy) incentive compatible* if for each firm truth-telling ($\tilde{f}_i = f_i$) is an optimal strategy to minimize his/her net cost regardless of the other agents' reporting strategy.

B. Cost function bidding vs. supply function bidding

The cost function bidding model described above is closely related to the supply function bidding models [10]. In a simple case where a production firm i is producing a single commodity (e.g., electricity), a supply function $s_i(p)$ describes the desired level of production as a function of price p at which the product can be exchanged in the market. Mathematically, a production cost function $f_i(x_i)$ and a supply function $s_i(p)$ are related by $s_i(p) = \text{argmax}_{x_i} \, px_i - f_i(x_i)$. For consuming firms, there is a similar relationship between utility functions and demand functions. From this relationship (c.f., Legendre transformation), it can be seen that a cost function bidding model is equivalent to supply function bidding model under mild assumptions (e.g., convexity of f_i). For instance, for $a > 0$, $f_i(x_i) = ax_i^2$ if and only if $s_i(p) = \frac{1}{2a}p$. Thus, assuming a linear supply function is equivalent to assuming a quadratic cost function. Alternatively, suppose that $\mathcal{F}_i$ is the set of non-decreasing, continuously differentiable, and strongly convex functions from $[0, \infty)$ to $[0, \infty)$ such that $f_i(0) = 0$ and $f_i'(0) = 0$. Then $f_i \in \mathcal{F}_i$ if and only if $s_i \in \mathcal{S}_i$, where $\mathcal{S}_i$ is the set of non-decreasing continuous function from $[0, \infty)$ to $[0, \infty)$ such that $s_i(0) = 0$. For the rest of this paper, we focus on the cost function bidding model only.

III. Clearing-price and VCG mechanisms

In this section, we formally introduce the clearing-price mechanism and the VCG mechanism for the social cost minimization problem (1).

A. The clearing-price mechanism

The clearing-price mechanism is characterized by the following decision and payment rules.

Decision rule: Based on reported functions $\tilde{f}_i$, $i = 1, 2, ..., N$, the auctioneer formulates a problem $\tilde{\mathcal{P}}$, which is identical to $\mathcal{P}$ in (1) except that reported functions are used, and solve for the primal-dual optimal solution $(\tilde{x}^*, \tilde{\mu}^*, \tilde{\lambda}^*)$ satisfying the KKT condition:[2]

$$\nabla \tilde{f}_i(\tilde{x}_i^*) + \tilde{\mu}^{*\top} \nabla g_i(\tilde{x}_i^*) + \tilde{\lambda}^{*\top} \nabla h_i(\tilde{x}_i^*) = 0 \tag{2a}$$

$$\sum\nolimits_{i=1}^{N} g_i(\tilde{x}_i^*) \leq 0, \; \sum\nolimits_{i=1}^{N} h_i(\tilde{x}_i^*) = 0 \tag{2b}$$

$$\tilde{\mu}^{*\top} \sum\nolimits_{i=1}^{T} g_i(\tilde{x}_i^*) = 0 \tag{2c}$$

$$\tilde{\mu}^* \geq 0, \tag{2d}$$

[1] We assume x is a column vector. We write $(x; y) = [x^\top \; y^\top]^\top$.

[2] When a function $f : \mathbb{R}^N \to \mathbb{R}^k$ is differentiable at $x \in \mathbb{R}^N$, we define the gradient matrix of f at x as $\nabla f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_N} \end{bmatrix}$.

where (2a) is imposed for each $i = 1, ..., N$. A social decision is determined by $\tilde{x}^* = (\tilde{x}_1^*; ...; \tilde{x}_N^*)$.

Payment rule: The auctioneer computes a vector of the "market clearing prices"

$$\tilde{p}_i^* = \tilde{\mu}^{*\top} \nabla g_i(\tilde{x}_i^*) + \tilde{\lambda}^{*\top} \nabla h_i(\tilde{x}_i^*)$$

for each $i = 1, ..., N$. The payment (from the firm i to the auctioneer) is determined by

$$\pi_i^{\text{clearing}} = \tilde{p}_i^* \tilde{x}_i^*. \tag{3}$$

The clearing-price mechanism and its variants are widely used in practice of power system operations because of its simplicity. It is also used as an underlying market model in the existing market analyses (e.g., [6]–[9]). A drawback of this mechanism is that sometimes the existence and uniqueness of the solution satisfying the KKT condition are not guaranteed especially when a nonconvex problem formulation is used [21]. Also, the clearing-price mechanism is not incentive compatible in general, as demonstrated in the next simple example.

Example 1: Consider a model of power supply market with N power generating firms. Let x_i be the production by the i-th firm, and suppose that the total power supply must meet the total demand N. Assume that the true generation cost function is given by $f_i(x_i) = x_i^2$ for every $i = 1, 2, ..., N$. The optimal allocation x that minimizes the social cost is obtained by solving

$$\min_x \quad \sum_{i=1}^N f_i(x_i) \tag{4a}$$

$$\text{s.t.} \quad \sum_{i=1}^N x_i = N. \tag{4b}$$

Clearly, the optimal solution is $x^* = (1; 1; \cdots; 1)$. Suppose that the function $f_i(x_i)$ is publicly unknown besides the fact that it belongs to the set $\mathcal{F}_i = \{a_i x_i^2 : a_i > 0\}$. Thus, each firm i is required to report his/her private parameter a_i to the auctioneer. $\square$

Suppose that the clearing-price mechanism is applied to Example 1. Based on the reported functions $\tilde{f}_i(x_i) = a_i x_i^2$, the auctioneer solves the KKT condition

$$\nabla_{x_i} \tilde{f}_i(\tilde{x}_i^*) + \tilde{\lambda}^* = 0 \quad \forall i = 1, 2, \cdots, N \tag{5a}$$

$$\sum_{i=1}^N \tilde{x}_i^* = N \tag{5b}$$

for a social decision $\tilde{x}^*$ and the "market-clearing price" $\tilde{\lambda}^*$. The auctioneer computes the payments for individual firms by $\pi_i^{\text{clearing}} = \tilde{\lambda}^* \tilde{x}_i^*$. (Note that π_i^{clearing} is negative in this example because we follow the convention that the payment is from the firm to the auctioneer by default.) Assuming all generators submit their cost functions truthfully (i.e., $\tilde{f}_i(x_i) = x_i^2$), then $\tilde{x}^* = (1; 1; \cdots; 1)$ and $\tilde{\lambda}^* = 2$ satisfy the KKT condition. Hence, the reward that each generator receives is calculated as $-\tilde{\lambda}^* \tilde{x}_i^* = 2$, and the net cost for the firm i is $C_i = f_i(\tilde{x}_i^*) + \pi_i^{\text{clearing}} = 1 - 2 = -1$ (profit is $+1$).

To demonstrate that the clearing-price mechanism is not incentive compatible, suppose that the first firm misreports

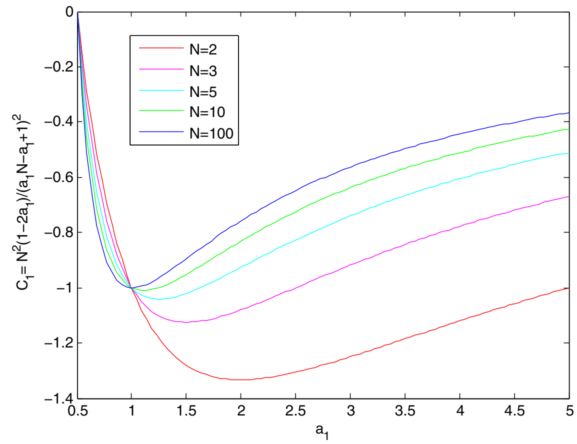

Fig. 1. The first generator's report a_1 and the resulting cost C_1.

($a_1 \neq 1$), while other firms remain truthful ($a_i = 1$, $i = 2, 3, \cdots, N$). The KKT condition (5) is

$$2a_1 \tilde{x}_1^* + \tilde{\lambda}^* = 0, \quad 2\tilde{x}_i^* + \tilde{\lambda}^* = 0 \quad \forall i = 2, 3, \cdots, N$$
$$\sum_{i=1}^N \tilde{x}_i^* = N$$

from which we have

$$\tilde{\lambda}^* = -\frac{2a_1 N}{a_1 N - a_1 + N}, \quad \tilde{x}_1^* = \frac{N}{a_1 N - a_1 + N}$$
$$\tilde{x}_i^* = \frac{a_1 N}{a_1 N - a_1 + N} \quad \forall i = 2, 3, \cdots, N.$$

Now, the net cost for the first firm is calculated as

$$C_1 = f_1(\tilde{x}_1^*) + \pi_1^{\text{clearing}} = \frac{N^2(1 - 2a_1)}{(a_1 N - a_1 + 1)^2}.$$

Figure 1 plots C_1 as a function of a_1 for various N. It can be seen that the first generator can achieve the minimum net cost $C_1 = -1 - \frac{1}{N^2-1}$ by reporting $a_1 = 1 + \frac{1}{N-1}$. The net cost in this case is smaller than that follows from the truthful report (which is -1). Thus we conclude that truthful report is not a Nash equilibrium (and thus not a dominant strategy). It is also worth mentioning that as the number of participants increases ($N \to +\infty$), the optimal reporting strategy tends to be the truthful one ($a_1 \to 1$).

If all participants are truthful in the clearing-price mechanism, the payment for the i-th participant is

$$\pi_i^{\text{clearing, truthful}} = p_i^* x_i^* \tag{6}$$

with $p_i^* = \mu^{*\top} \nabla g_i(x_i^*) + \lambda^{*\top} \nabla h_i(x_i^*)$ where (x^*, μ^*, λ^*) is the primal-dual optimal solution to $\mathcal{P}$.

B. The VCG mechanism

The VCG mechanism, also known as the pivot mechanism, chooses the payment rule in such a way that the i-th firm's payment is equal to its externality. To this end, the mechanism evaluates the impact of each firm's "presence" in the market by introducing the following auxiliary optimization problem for each $i = 1, 2, \cdots, N$:

$$\mathcal{P}^{-i}(x_i) : \min_{x_{-i}} f(x_i, x_{-i}) \triangleq \sum_{j \neq i}^{N} f_j(x_j) + f_i(x_i) \tag{7a}$$

$$\text{s.t. } g(x_i, x_{-i}) \triangleq \sum_{j \neq i}^{N} g_j(x_j) + g_i(x_i) \leq 0 \tag{7b}$$

$$h(x_i, x_{-i}) \triangleq \sum_{j \neq i}^{N} h_j(x_j) + h_i(x_i) = 0. \tag{7c}$$

Decision variables in (7) are $x_{-i} = (x_1; ...; x_{i-1}; x_{i+1}; ...; x_N)$. Notice that in $\mathcal{P}^{-i}(x_i)$, an optimal decision x_{-i} for the firms excluding i needs to be found while x_i is fixed. Denote by

$$x^{-i*}(x_i) = \left(x_1^{-i*}(x_i); ...; x_{i-1}^{-i*}(x_i); x_{i+1}^{-i*}(x_i); ...; x_N^{-i*}(x_i)\right)$$

an optimal solution to the problem $\mathcal{P}^{-i}(x_i)$. Note that $x^{-i*}(x_i)$ is parametrized by x_i. We interpret $x^{-i*}(0)$ as an optimal social decision for the firms other than i when the i-th firm is absent from the market.[3] The VCG mechanism is characterized by the following decision and payment rules.

Decision rule: Based on the reported functions $\tilde{f}_i$, $i = 1, 2, ..., N$, the auctioneer formulates optimization problems $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}^{(-i)}(0)$ $\forall i = 1, \cdots, N$, which are identical to $\mathcal{P}$ and $\mathcal{P}^{-i}(0)$ except that reported functions are used. The auctioneer determines a social decision $\tilde{x}^*$ as an optimal solution to $\tilde{\mathcal{P}}$. The auctioneer also computes an optimal solution $\tilde{x}^{-i*}(0)$ to $\tilde{\mathcal{P}}^{-i}(0)$ for every $i = 1, \cdots, N$.

Payment rule: Payment for the firm i is calculated by

$$\pi_i^{\text{VCG}} = \sum\nolimits_{j \neq i} \tilde{f}_j(\tilde{x}_j^*) - \sum\nolimits_{j \neq i} \tilde{f}_j(\tilde{x}_j^{-i*}(0)). \tag{8}$$

The next Theorem is a straightforward application of the well-known result (e.g., [2, Ch. 23] [19, Ch.10] [23]) to the social cost minimization problem (1).

Theorem 1: The VCG mechanism is dominant strategy incentive compatible.

Proof: Although this is a minor variation of the standard results, we provide a proof for the sake of completeness. We first assume that there exists an index i and reporting strategies $\tilde{f}_j$ for $j \neq i$ such that the firm i attains a smaller net cost C_i by misreporting (i.e., $\tilde{f}_i \neq f_i$), and then derive a contradiction.

Denote by $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^{-i}(x_i)$ the problems (1) and (7) when functions $(\tilde{f}_1, ..., \tilde{f}_{i-1}, f_i, \tilde{f}_{i+1}, ..., \tilde{f}_N)$ are reported. Let $\hat{x}^*$ and $\hat{x}^{-i*}(x_i)$ be the optimal solutions to $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^{-i}(x_i)$ respectively. The net cost for the i-th firm is

$$\begin{aligned}\hat{C}_i &= f_i(\hat{x}_i^*) + \hat{\pi}_i^{\text{VCG}} \\ &= f_i(\hat{x}_i^*) + \sum\nolimits_{j \neq i} \tilde{f}_j(\hat{x}_j^*) - \sum\nolimits_{j \neq i} \tilde{f}_j(\hat{x}_j^{-i*}(0)).\end{aligned}$$

Also, denote by $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}^{-i}(x_i)$ the problems (1) and (7) when functions $(\tilde{f}_1, ..., \tilde{f}_{i-1}, \tilde{f}_i, \tilde{f}_{i+1}, ..., \tilde{f}_N)$ are reported. Let $\tilde{x}^*$ and $\tilde{x}^{-i*}(x_i)$ be the optimal solutions to $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}^{-i}(x_i)$ respectively. The net cost in this case is

$$\begin{aligned}\tilde{C}_i &= f_i(\tilde{x}_i^*) + \tilde{\pi}_i^{\text{VCG}} \\ &= f_i(\tilde{x}_i^*) + \sum\nolimits_{j \neq i} \tilde{f}_j(\tilde{x}_j^*) - \sum\nolimits_{j \neq i} \tilde{f}_j(\tilde{x}_j^{-i*}(0)).\end{aligned}$$

Now, notice that $\hat{x}_j^{-i*}(0) = \tilde{x}_j^{-i*}(0)$, i.e., the i-th firm's reporting strategy does not alter the solution to the auxiliary problem. Thus $\tilde{C}_i < \hat{C}_i$ implies

$$f_i(\tilde{x}_i^*) + \sum\nolimits_{j \neq i} \tilde{f}_j(\tilde{x}_j^*) < f_i(\hat{x}_i^*) + \sum\nolimits_{j \neq i} \tilde{f}_j(\hat{x}_j^*).$$

However, this is a contradiction to our hypothesis that $\hat{x}^*$ is an optimal solution to $\hat{\mathcal{P}}$. ■

In view of Theorem 1, the VCG mechanism is said to *implement an efficient social choice function in dominant strategies* [23]. Assuming all reports are truthful, the VCG payment becomes[4]

$$\pi_i^{\text{VCG}} = \sum\nolimits_{j \neq i} f_j(x_j^*) - \sum\nolimits_{j \neq i} f_j(x_j^{-i*}(0)). \tag{9}$$

It is instrumental to notice the difference between the VCG payment (9) and that of the clearing-price mechanism (6) with truthful participants. To quantify the difference, consider an application of the VCG mechanism to the scenario in Example 1. For each $i = 1, \cdots, N$, the VCG payment is computed as

$$\begin{aligned}\pi_i^{\text{VCG}} &= \sum\nolimits_{j \neq i} f_j(x_j^*) - \sum\nolimits_{j \neq i} f_j(x_j^{-i*}(0)) \\ &= \sum\nolimits_{j \neq i} f_j(1) - \sum\nolimits_{j \neq i} f_j(\frac{N}{N-1}) \\ &= -2 - \frac{1}{N-1}\end{aligned} \tag{10}$$

In other words, each firm receives $2 + \frac{1}{N-1}$ for producing $x_i = 1$. Recall that the payment determined by the clearing-price mechanism when all reports were truthful was 2. Therefore, in this example, we have $\pi_i^{\text{clearing, truthful}} > \pi_i^{\text{VCG}}$ for all i, even though in both cases the production by the firms are the same. Also, notice that the difference diminishes as $N \to +\infty$.

IV. Path Integral representations of VCG payments

In this section, we establish a mathematical relationship between the clearing-price and the VCG mechanisms in a general setting. We first recall some basic results from nonlinear programming. A feasible vector x_{-i} for the problem $\mathcal{P}^{-i}(x_i)$ is said to be *regular* [24] if the gradient matrix

$$\begin{bmatrix} \nabla_{x_{-i}} \tilde{g}(x_i, x_{-i}) \\ \nabla_{x_{-i}} h(x_i, x_{-i}) \end{bmatrix}$$

has linearly independent rows. Here, $\tilde{g}(x_i, x_{-i})$ is the collection of all active constraints in $g(x_i, x_{-i}) \leq 0$, i.e., the collection of indices k such that $g_k(x_i, x_{-i}) = 0$. If x^{-i*} is regular, there exists a unique set of Lagrange multipliers

[3] In this paper, we interpret the i-th firm being absent from the market as $x_i = 0$. See [22] for related discussions.

[4] We do not distinguish (8) and (9) since they are equal under a realistic assumption that all reports are truthful.

$\mu^{-i*} \in \mathbb{R}^k, \lambda^{-i*} \in \mathbb{R}^l$ such that the following KKT conditions are satisfied [24, Proposition 3.3.1].

$$\nabla_{x_j} f_j(x_j^{-i*}) + {\mu^{-i*}}^\top \nabla g_j(x_j^{-i*}) + {\lambda^{-i*}}^\top \nabla h_j(x_j^{-i*}) = 0 \;\; \forall j \neq i \tag{11a}$$

$${\mu^{-i*}}^\top \left(\sum\nolimits_{j\neq i} g_j(x_j^{-i*}) + g_i(x_i) \right) = 0 \tag{11b}$$

$$\mu^{-i*} \geq 0. \tag{11c}$$

For notational simplicity, in what follows, we often suppress the dependency of $(x^{-i*}, \mu^{-i*}, \lambda^{-i*})$ on x_i.

For each $i = 1, 2, \cdots, N$, consider a smooth path $\gamma_i : [0,1] \to \mathbb{R}^{n_i}$ such that $\gamma_i(0) = 0$ and $\gamma_i(1) = x_i^*$. Notice that the origin of the path corresponds to the case where the i-th firm is absent from the market ($x_i = 0$) and the terminal of the path corresponds to the case where the i-th firm is assigned an optimal solution to (1) ($x_i = x_i^*$).

For each point $x_i \in \gamma_i$ on the path, as before, we write $x^{-i*}(x_i)$ to denote an optimal solution to $\mathcal{P}^{-i}(x_i)$. Whenever $x^{-i*}(x_i)$ is regular, we also denote by $(\mu^{-i*}(x_i), \lambda^{-i*}(x_i))$ the corresponding unique set of Lagrange multipliers. Notice that for each $i = 1, ..., N$, $(x^{-i*}(x_i^*), \mu^{-i*}(x_i^*), \lambda^{-i*}(x_i)) = (x_{-i}^*, \mu^*, \lambda^*)$.

The next theorem shows that VCG payment π_i^{VCG} is equal to the path integral of the vector field of the market clearing prices along γ_i.

Theorem 2: Assume f, g and h are continuously differentiable, and $\mathcal{P}$ admits an optimal solution x^*. Let $\gamma_i : [0,1] \to \mathbb{R}^{n_i}$ be any smooth path such that $\gamma_i(0) = 0$, $\gamma_i(1) = x_i^*$, and the following conditions are satisfied:

(A) $\mathcal{P}^{-i}(\gamma_i(t))$ admits an optimal solution $x^{-i*}(\gamma_i(t))$ for every $t \in [0,1]$.

(B) $x^{-i*}(\gamma_i(t))$ is continuously differentiable in t, and is regular almost everywhere in $[0,1]$.

(C) $\mu^{-i*}(\gamma_i(t))$ and $\lambda^{-i*}(\gamma_i(t))$ are differentiable almost everywhere in $[0,1]$.

Then, the following equality holds:

$$\pi_i^{\text{VCG}} = \int_{\gamma_i} p_i^*(x_i) dx_i \left(= \int_0^1 p_i^*(\gamma_i(t)) \frac{d\gamma_i}{dt} dt \right) \tag{12}$$

where $p_i^*(x_i) = \mu^{-i*}(x_i)^\top \nabla g_i(x_i) + \lambda^{-i*}(x_i)^\top \nabla h_i(x_i)$.

Proof: Since $x^{-i*}(\gamma_i(t))$ is a feasible solution to $\mathcal{P}^{-i}(x_i)$, it must satisfy the equality constraint $\sum_{j\neq i} h_j(x_j^{-i*}) + h_i(\gamma_i(t)) = 0$ for every $t \in [0,1]$. Differentiating with respect to t, we obtain

$$\sum\nolimits_{j\neq i} \nabla h_j(x_j^{-i*}) \frac{dx_j^{-i*}}{dt} + \nabla h_i(\gamma_i(t)) \frac{d\gamma_i}{dt} = 0. \tag{13}$$

Next, since the left hand side of (11b) is differentiable at almost every $t \in [0,1]$,

$$\left(\frac{d\mu^{-i*}}{dt}\right)^\top \left(\sum_{j\neq i} g_j(x_j^{-i*}) + g_i(\gamma_i(t)) \right) + {\mu^{-i*}}^\top \left(\sum_{j\neq i} \nabla g_j(x_j^{-i*}) \frac{dx_j^{-i*}}{dt} + \nabla g_i(\gamma_i(t)) \frac{d\gamma_i}{dt} \right) = 0 \tag{14}$$

almost everywhere in $[0,1]$. We claim that the first term of (14) is zero. To see this, suppose that the ℓ-th component of $\sum_{j\neq i} g_j(x_j^{-i*}) + g_i(\gamma_i(t))$ is strictly negative at some $t \in [0,1]$. Due to continuity, it is always possible to choose an interval $[t-\epsilon, t+\epsilon]$ on which the ℓ-th component of $\sum_{j\neq i} g_j(x_j^{-i*}) + g_i(\gamma_i(t))$ is strictly negative. By the complementary slackness condition (11b), the ℓ-th component of $\mu^{-i*}(\gamma_i(t))$ is zero on this interval, and so is $\frac{d\mu^{-i*}}{dt}$. Thus (14) becomes

$${\mu^{-i*}}^\top \left(\sum_{j\neq i} \nabla g_j(x_j^{-i*}) \frac{dx_j^{-i*}}{dt} + \nabla g_i(\gamma_i(t)) \frac{d\gamma_i}{dt} \right) = 0 \tag{15}$$

almost everywhere in $[0,1]$. Now, the right hand side of (12) becomes

$$\begin{aligned}
&\int_0^1 \left[\mu^{-i*}(\gamma_i(t))^\top \nabla g_i(\gamma_i(t)) + \lambda^{-i*}(\gamma_i(t))^\top \nabla h_i(\gamma_i(t))\right] \frac{d\gamma_i}{dt} dt \\
&= -\sum_{j\neq i} \int_0^1 \left[\mu^{-i*}(\gamma_i(t))^\top \nabla g_j(x_j^{-i*}) + \lambda^{-i*}(\gamma_i(t))^\top \nabla h_j(x_j^{-i*})\right] \frac{dx_j^{-i*}}{dt} dt && (16) \\
&= \sum\nolimits_{j\neq i} \int_0^1 \nabla f_j(x_j^{-i*}) \frac{dx_j^{-i*}}{dt} dt && (17) \\
&= \sum\nolimits_{j\neq i} f_j(x_j^{-i*}(x_i^*)) - \sum\nolimits_{j\neq i} f_j(x_j^{-i*}(0)) && (18) \\
&= \sum\nolimits_{j\neq i} f_j(x_j^*) - \sum\nolimits_{j\neq i} f_j(x_j^{-i*}(0)) && (19) \\
&= \pi_j^{\text{VCG}}. && (20)
\end{aligned}$$

In (16), we have used (13) and (15). In (17), the KKT condition (11a) was used. Since $\nabla f_j(x_j^{-i*}) \frac{dx_j^{-i*}}{dt}$ is continuous in t, the fundamental theorem of calculus is applicable in (18). ■

Remark 2: A special case of (12) can be found in [2, Ch. 23.C]. Theorem 2 also extends [25, Proposition 4] to general class of problems of the form $\mathcal{P}$.

A. Example: Simple power supply market

We consider again the power supply market in Example 1. Recall that $x_i^* = 1$ for every i is the optimal solution. So define a smooth path $\gamma_i : [0,1] \to \mathbb{R}$ by $\gamma_i(t) = t$ so that $\gamma_i(0) = 0$ and $\gamma_i(1) = x_i^*$. Since $h_i(x_i) = x_i$, we have $\nabla_{x_i} h_i(x_i) = 1$. There is no inequality constraint. Therefore, (12) becomes

$$\pi_i^{\text{VCG}} = \int_0^1 \lambda^{-i*}(x_i) dx_i. \tag{21}$$

Equation (21) can also be verified as follows. Note that $\mathcal{P}^{-i}(x_i)$ is

$$\begin{aligned}
\min_{x_{-i}} \;\; & \sum\nolimits_{j\neq i}^N x_j^2 + x_i^2 \\
\text{s.t.} \;\; & \sum\nolimits_{j\neq i}^N x_j = N - x_i.
\end{aligned}$$

It is easy to verify that the optimal solution is regular and the corresponding Lagrange multiplier is $\lambda^{-i*}(x_i) = \frac{2(x_i-N)}{N-1}$. Thus the right hand side of (21) is computed as

$$\int_0^1 \lambda^{-i*}(x_i)dx_i = \int_0^1 \frac{2(x_i-N)}{N-1}dx_i = -2 - \frac{1}{N-1}.$$

This result coincides with π_i^{VCG} obtained in (10).

B. Graphical interpretation of Theorem 2

In this subsection, we introduce a graphical interpretation of the formula (12) for the cases with $n_i = 1$. In Figure 2, let $p_i^*(x_i)$ (defined in Theorem 2) be the market clearing price written as a function of x_i. In particular, $p_i^*(0)$ is the market clearing price when the i-th firm's consumption (production) is zero and $p_i^* = p_i^*(x_i^*)$ is the market clearing price when it is x_i^* (i.e., the optimal solution to $\mathcal{P}$). The formula (12) shows that the VCG tax (the monetary payment from the firm to the auctioneer) is equal to the area below the curve of $p_i^*(x_i)$. On the other hand, by definition (6), the tax imposed by the clearing-price mechanism *when all reports by the firms are truthful* is equal to the area of the rectangle shown in Figure 2.

Figure 2 shows that the discrepancy between π_i^{VCG} and $\pi_i^{\text{clearing, truthful}}$ is closely related to the market power of the i-th firm. The next result holds for general cases with $n_i \geq 1$.

Lemma 1: Let the optimization problem $\mathcal{P}$ and the smooth path $\gamma_i : [0,1] \to \mathbb{R}^n$ defined by $\gamma_i(t) = x_i^* t$ satisfy the conditions in Theorem 2. Suppose, in addition, that there exists a constant $\epsilon > 0$ such that

$$\|p_i^*(x_i) - p_i^*\| \leq \epsilon \tag{22}$$

for each point x_i on γ_i, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^{n_i}$. Then $|\pi_i^{\text{VCG}} - \pi_i^{\text{clearing, truthful}}| \leq \epsilon\|x_i^*\|$.

Proof: Due to the Cauchy-Schwarz inequality, we have

$$-\epsilon\|x_i^*\| \leq p_i^*(\gamma_i(t))x_i^* - p_i^* x_i^* \leq \epsilon\|x_i^*\|$$

for all $t \in [0,1]$. Integrating each side with respect to t, and using the formula (12):

$$\pi_i^{\text{VCG}} = \int_0^1 p_i^*(\gamma_i(t))\frac{d\gamma_i}{dt}dt = \int_0^1 p_i^*(\gamma_i(t))x_i^* dt,$$

we obtain $-\epsilon\|x_i^*\| \leq \pi_i^{\text{VCG}} - \pi_i^{\text{clearing, truthful}} \leq \epsilon\|x_i^*\|$. ■

Lemma 1 shows that if the price markup by firm i is bounded in the sense of (22) (graphically, the function $p_i^*(x_i)$ in Figure 2 is nearly constant), then π_i^{VCG} is close to $\pi_i^{\text{clearing, truthful}}$.

V. Budget loss analysis

Although the VCG mechanism implements efficient social choice functions in dominant strategies, budget balance property is not guaranteed in general. This is a prominent reason why the VCG mechanism is rarely used in practice. In this section, we apply Theorem 2 to analyze the budget loss of the VCG mechanism. Let $B_{\text{loss}}^{\text{VCG}} \triangleq -\sum_{i=1}^N \pi_i^{\text{VCG}}$ be the budget loss in the VCG mechanism, and $B_{\text{loss}}^{\text{clearing, truthful}} \triangleq -\sum_{i=1}^N \pi_i^{\text{clearing, truthful}}$ be the budget loss in the clearing-price mechanism *under the assumption that all players are truthful*.

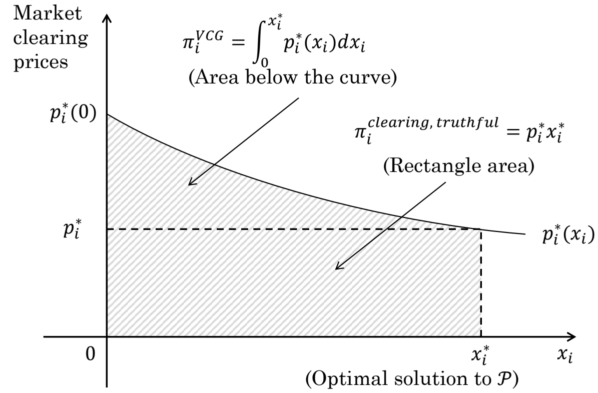

Fig. 2. VCG vs clearing-price mechanisms.

A. Quadratic cost functions

Consider a special case of (1) in which each cost function is of the form $f_i(x_i) = a_i x_i^2$ with $a_i > 0$ and the only constraint is $\sum_{i=1}^N x_i = c$. The next result shows that the quantity $(B_{\text{loss}}^{\text{VCG}} - B_{\text{loss}}^{\text{clearing, truthful}})/B_{\text{loss}}^{\text{clearing, truthful}}$ can be explicitly computed in terms of individual firm's share $s_i \triangleq \frac{x_i^*}{c}$.

Theorem 3: For each $i = 1, ..., N$, assume that the cost function has the form $f_i(x_i) = a_i x_i^2$ with $a_i > 0$ and the only constraint is $\sum_{i=1}^N x_i = c$. Then

$$\frac{B_{\text{loss}}^{\text{VCG}} - B_{\text{loss}}^{\text{clearing, truthful}}}{B_{\text{loss}}^{\text{clearing, truthful}}} = \sum_{i=1}^N \frac{s_i^2}{2(1-s_i)}.$$

Proof: From the KKT condition, for each $i = 1, ..., N$, we have $\lambda^{-i*}(x_i) = 2\left(\sum_{j\neq i}\frac{1}{a_j}\right)^{-1}(x_i - c)$. Thus

$$\pi_i^{\text{VCG}} = \int_0^{x_i^*} \lambda^{-i*}(x_i)dx_i = 2\left(\sum_{j\neq i}\frac{1}{a_j}\right)^{-1}\left(\frac{1}{2}{x_i^*}^2 - cx_i^*\right).$$

From the KKT condition, we also have $\frac{1}{a_i} = -\frac{2x_i^*}{\lambda^*}$. Substituting this into the above equality, we have $\pi_i^{\text{VCG}} = \lambda^* x_i^* - \frac{\lambda^* {x_i^*}^2}{2(x_i^*-c)}$. Thus,

$$\begin{aligned}\frac{B_{\text{loss}}^{\text{VCG}} - B_{\text{loss}}^{\text{clearing}}}{B_{\text{loss}}^{\text{clearing}}} &= \left(-\sum_{i=1}^N \pi_i^{\text{VCG}} + \lambda^* c\right)/(-\lambda^* c) \\ &= \sum_{i=1}^N \frac{x_i^*/c}{2(1-x_i^*/c)}\frac{x_i^*}{c} \\ &= \sum_{i=1}^N \frac{s_i^2}{2(1-s_i)}.\end{aligned}$$

■

In particular, if $s_i = 1/N$ for all i, we have $(B_{\text{loss}}^{\text{VCG}} - B_{\text{loss}}^{\text{clearing, truthful}})/B_{\text{loss}}^{\text{clearing, truthful}} \to 0$ as $N \to \infty$.

Remark 3: Although Theorem 3 shows $B_{\text{loss}}^{\text{VCG}} > B_{\text{loss}}^{\text{clearing, truthful}}$, it should not be concluded that the budget loss in the VCG mechanism is larger than that of the clearing-price mechanism. This is because $B_{\text{loss}}^{\text{clearing, truthful}}$ is evaluated with the assumption that all players are truthful, and does not reflect the realistic budget loss when the players are strategic in the clearing-price mechanism.

B. Bounded market power

In general, it is difficult to provide a bound on the budget loss of the VCG mechanism. Here, we present a direct consequence of Lemma 1.

Theorem 4: Suppose that $\mathcal{P}$ and the smooth path $\gamma_i : [0,1] \to \mathbb{R}^{n_i}$ defined by $\gamma_i(t) = x_i^* t$ satisfy the conditions in Theorem 2 for each $i = 1, 2, ..., N$. Suppose there exists a constant $\epsilon > 0$ such that $\|p_i^*(x_i) - p_i^*\| \leq \epsilon$ for every point x_i on γ_i and for each i. Then

$$|B_{\text{loss}}^{\text{VCG}} - B_{\text{loss}}^{\text{clearing, truthful}}| \leq \epsilon \sum_{i=1}^{N} \|x_i^*\|.$$

Proof: By Lemma 1, we have

$$-\epsilon\|x_i^*\| \leq \pi_i^{\text{VCG}} - \pi_i^{\text{clearing, truthful}} \leq \epsilon\|x_i^*\|$$

for each i. Thus the claim follows immediately. ∎

Theorem 4 can be applied to the following supply-demand matching problem with transmission constraints:

$$\min_{s,d} \ \sum_{i\in\mathcal{S}} c_i(s_i) - \sum_{i\in\mathcal{D}} u_i(d_i) \tag{23a}$$

$$\text{s.t.} \ \sum_{i\in\mathcal{S}} s_i = \sum_{i\in\mathcal{D}} d_i, \ s \geq 0, d \geq 0 \tag{23b}$$

$$-b \leq H^G s - H^L d \leq b. \tag{23c}$$

In (23), $\mathcal{S}$ is the set of generators, $\mathcal{D}$ is the set of consumers, $c_i, i \in \mathcal{S}$ are generation cost functions, $u_i, i \in \mathcal{D}$ are demand utility functions, H^G is the generation shift factor matrix, H^L is the load shift factor matrix, and b is the transmission constraint. With $x = (s, -d)$, the optimization problem (23) can be written in the standard form (1).

Notice that the quantity $\sum_{i=1}^{N} \|x_i^*\|$ in Theorem 4 is twice the total generation (consumption) in the system (23). Therefore, if the premises of Theorem 4 are satisfied, it is concluded from Theorem 4 that the excess budget loss of the VCG mechanism over the clearing-price mechanism with truthful participants is upper bounded in terms of the bound ϵ on the price markup and the total generation (consumption) in the system.

VI. Conclusion

In this paper, we considered a fundamental connection between the clearing-price mechanism (which is commonly used in the efficiency loss analysis) and the VCG mechanism. As the main technical result, we showed that the VCG payment (tax) is equal to the path integral of the vector fields of market clearing prices, from which we showed that the clearing-price mechanism and the VCG mechanism have similar tax rules when individual participants' market power is negligible. We also outlined how this connection can be exploited to analyze the budget loss of the VCG mechanism. Although this study is motivated by the electricity market design, the results of this paper are general and can be applied to other market design problems.

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