On the Exponential Stability of Primal-Dual Gradient Dynamics*

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Abstract—Continuous time primal-dual gradient dynamics that find a saddle point of a Lagrangian of an optimization problem have been widely used in systems and control. While the global asymptotic stability of such dynamics has been well-studied, it is less studied whether they are globally exponentially stable. In this paper, we study the primal-dual gradient dynamics for convex optimization with strongly-convex and smooth objectives and affine equality or inequality constraints, and prove global exponential stability for such dynamics. Bounds on decaying rates are provided.

I. INTRODUCTION

This paper considers the following constrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad A_1 x = b_1, \quad A_2 x \leq b_2$$

where $A_1 \in \mathbb{R}^{m_1 \times n}, A_2 \in \mathbb{R}^{m_2 \times n}$ and $b_1 \in \mathbb{R}^{m_1}, b_2 \in \mathbb{R}^{m_2}$ and $f(x)$ is a strongly convex and smooth function. Let $L(x, \lambda)$ be the Lagrangian (or Augmented Lagrangian) associated with Problem (1). The focus of this paper is the following primal-dual gradient dynamics, also known as saddle-point dynamics, associated with the Lagrangian $L(x, \lambda)$,

$$\dot{x} = -\eta_1 \nabla_x L(x, \lambda) \quad (2a)$$
$$\dot{\lambda} = -\eta_2 \nabla_{\lambda} L(x, \lambda) \quad (2b)$$

where $\eta_1, \eta_2 > 0$ are time constants.

Primal-Dual Gradient Dynamics (PDGD), also known as saddle-point dynamics, were first introduced in [1], [2]. They have been widely used in engineering and control systems, for example in power grid [3], [4], wireless communication [5], [6], network and distributed optimization [7], [8], game theory [9], etc. Despite its wide applications, general studies on PDGD [1], [2], [8], [10]–[22] have mostly focused on its asymptotic stability (or convergence), with few studying its global exponential stability. It is known that the gradient dynamics for the unconstrained version of (1) achieves global exponential stability when $f$ is strongly convex and smooth. It is natural to raise the question whether in the constrained case, PDGD can also achieve global exponential stability.

Global exponential stability is a desirable property in practice. Firstly, in control systems especially those in critical infrastructure like the power grid, it is desirable to have strong stability guarantees. Secondly, when using PDGD as computational tools for constrained optimization, discretization is essential for implementation. The global exponential stability ensures that the simple explicit Euler discretization has a geometric convergence rate when the discretization step size is sufficiently small [23], [24]. This is an appealing property for discrete-time optimization methods.

Contribution of this paper. In this paper, we prove the global exponential stability of PDGD (2) under some regularity conditions on problem (1) and we also give bounds on the decaying rates (Theorem 1 and 2). Our proof relies on a quadratic Lyapunov function that has non-zero off-diagonal terms, which is different from the (block-)diagonal quadratic Lyapunov functions that are commonly used in the literature [6], [20] and are known being unable to certify global exponential stability [6, Lemma 3]. We also highlight that when handling inequality constraints, we use a variant of the PDGD based on Augmented Lagrangian [25] and is projection free. This is different from the projection-based PDGD studied in [18], [20], which is discontinuous (see Footnote 2 for more discussions). Our variant of PDGD guarantees that the multipliers stay nonnegative without using projection, and avoids the discontinuity problem caused by projection [18], [20] (see Remark 1 for more discussions).

A. Related Work

There have been many efforts in studying the stability of PDGD as well as its discrete time version. An incomplete list includes [1], [2], [8], [10]–[22]. For instance, [17] studies the subgradient saddle point algorithm and proves its convergence to an approximate saddle point with rate $O(\frac{1}{t})$; [18] uses LaSalle invariance principle to prove global asymptotic stability of PDGD; [22] studies the global asymptotic stability of the saddle-point dynamics associated with general saddle functions; and [20] proves global asymptotic stability of PDGD with projection, which is extended in [21] by using a weaker assumption and proving input-to-state stability.

Our work is closely related to a recent paper [8] which studies saddle-point-like dynamics and proves global exponential stability when applying such dynamics to equality constrained convex optimization problems. The difference between [8] and our work is that for the equality constrained case, [8] considers a different Lagrangian from ours. Further, the result in [8] cannot be directly generalized to inequality constrained case [8, Remark 3.9].

Our work is also related to the vast literature on spectral bounds on saddle matrices [26], [27]. Such bounds, when combined with Ostrowski Theorem [28, 10.1.4], can lead to local exponential stability results of PDGD [25, Sec 4.4.1] [29, Prop. 4.4.1] as opposed to global exponential stability which is the focus of this paper.

It recently came to our attention that [30] studies a class of dynamics, a special case of which turns out to

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be similar to our PDGD for affine inequality constraints. [30] also proves global exponential stability. Their proof uses frequency domain analysis, which is different from our time-domain analysis. It remains interesting to investigate the connection between the methods of [30] and our work.

Notations. Throughout the paper, scalars will be small letters, vectors will be bold small letters and matrices will be capital letters. Notation $\| \cdot \|$ represents Euclidean norm for vectors, and spectrum norm for matrices. For any symmetric matrix $P_1, P_2$ of the same dimension, $P_1 \succeq P_2$ means $P_1 - P_2$ is positive semi-definite.

II. ALGORITHMS AND MAIN RESULTS

In this section we describe our PDGD for solving Problem (1) and present stability results. Throughout this paper, we use the following assumption of $f$:

**Assumption 1.** Function $f$ is twice continuously differentiable, $\mu$-strongly convex and $\ell$-smooth, i.e. for all $x, y \in \mathbb{R}^n$, 

$$\mu \|x - y\|^2 \leq \langle \nabla f(x) - \nabla f(y), x - y \rangle \leq \ell \|x - y\|^2 \quad (3)$$

To streamline exposition, we will present the equality constrained case and the inequality constrained case separately. Integrating them will give PDGD with global exponential stability for Problem (1). Without causing any confusion, notations will be double-used in the two cases.

A. Equality Constrained Case

We first consider the equality constrained case,

$$\min_{x \in \mathbb{R}^n} \ f(x) \quad (4)$$

s.t. $Ax = b$

Here we remove the subscript for $A$ and $b$ in Problem (1) for notational simplicity. Problem (4) has the Lagrangian,

$$L(x, \lambda) = f(x) + \lambda^T (Ax - b) \quad (5)$$

where $\lambda \in \mathbb{R}^m$ is the Lagrangian multiplier. The PDGD is,

$$\dot{x} = -\nabla_x L(x, \lambda) = -\nabla f(x) - A^T \lambda \quad (6a)$$

$$\dot{\lambda} = \eta \nabla_\lambda L(x, \lambda) = \eta (Ax - b) \quad (6b)$$

where without loss of generality, we have fixed the time constant of the primal part to be 1. We make the following assumption on $A$, which is the linear independence constraint qualification for (4).

**Assumption 2.** We assume that matrix $A$ is full row rank and $A^T A \preceq \kappa_1 I$ for some $\kappa_1, \kappa_2 > 0$.

Let $(x^*, \lambda^*)$ be the equilibrium point of (6), which in this case is also the saddle point of $L$. The following theorem gives the global exponential stability of the PDGD (6).

**Theorem 1.** Under Assumption 1 and 2, for $\eta > 0$, define $\tau_{eq} = \min\{\frac{\mu}{2\ell}, \frac{\mu}{\kappa_1}\}$. Then there exist constants $C_1, C_2$ that depend on $\eta, \kappa_1, \kappa_2, \mu, \ell, \|x(0) - x^*\|, \|\lambda(0) - \lambda^*\|$. s.t. $\|x(t) - x^*\| \leq C_1 e^{-\frac{\tau_{eq}}{2} t}$ and $\|\lambda(t) - \lambda^*\| \leq C_2 e^{-\frac{\tau_{eq}}{2} t}$.

1Assumption 1 and 2 guarantee that the saddle point exists and is unique.

B. Inequality Constrained Case

Now we consider the inequality constrained case,

$$\min_{x \in \mathbb{R}^n} \ f(x) \quad (7)$$

s.t. $Ax \preceq b$

where $f$ and $A$ satisfy Assumption 1 and 2. For the inequality constrained case, we use the "Augmented Lagrangian" [25, Sec. 3.1], as opposed to the standard Lagrangian in [18], [20]. In details, let $A^T = [a_1, a_2, \ldots, a_m]$, with each $a_j \in \mathbb{R}^n$, and let $b = [b_1, \ldots, b_m]^T$. Then we define the augmented Lagrangian,

$$L(x, \lambda) = f(x) + \sum_{j=1}^m H_\rho(a_j^T x - b_j, \lambda_j) \quad (8)$$

where $\rho > 0$ is a free parameter, $H_\rho(\cdot, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a penalty function on constraint violation, defined as follows

$$H_\rho(a_j^T x - b_j, \lambda_j) = \begin{cases} \frac{1}{2} \rho (a_j^T x - b_j)^2 & \text{if } \rho (a_j^T x - b_j) + \lambda_j \geq 0 \\ -\frac{\lambda_j^2}{2 \rho} & \text{if } \rho (a_j^T x - b_j) + \lambda_j < 0 \end{cases}$$

We can then calculate the gradient of $H_\rho$ w.r.t. $x$ and $\lambda$.

$$\nabla_x H_\rho(a_j^T x - b_j, \lambda_j) = \max\{\rho (a_j^T x - b_j) + \lambda_j, 0\} a_j$$

$$\nabla_\lambda H_\rho(a_j^T x - b_j, \lambda_j) = \max\{\rho (a_j^T x - b_j) + \lambda_j, 0\} - \lambda_j e_j$$

where $e_j \in \mathbb{R}^m$ is a vector with the $j$’th entry being 1 and all others being 0. The primal-dual gradient dynamics for the augmented Lagrangian $L$ is given in (9). We call it as Aug-PDGD (Augmented Primal-Dual Gradient Dynamics).

$$\dot{x} = -\nabla_x L(x, \lambda) = -\nabla f(x) - \sum_{j=1}^m \nabla_x H_\rho(a_j^T x - b_j, \lambda_j)$$

$$\dot{\lambda} = \eta \nabla_\lambda L(x, \lambda) = \eta (Ax - b)$$

**Remark 1.** It is easy to check that, if $\lambda_j(0) \geq 0$, then (9) guarantees $\lambda_j(t) \geq 0, \forall t$. This means that the dynamics (9) automatically guarantees $\lambda_j(t)$ will stay nonnegative as long as its initial value is nonnegative, without using projection as is done in [18], [20], thus avoiding discontinuity issues caused by the projection step.

Since the saddle point of the Augmented Lagrangian (8)
is the same as that of the standard Lagrangian (see [25, Sec. 3.1] for details), we have the following proposition regarding the equilibrium point \((x^*, \lambda^*)\) of Aug-PDGD. For completeness we include a proof in our online report [31, Appendix-E].

**Proposition 1 ( [25]).** Under Assumption 1 and 2, Aug-PDGD (9) has a unique equilibrium point \((x^*, \lambda^*)\) and it satisfies the KKT condition of problem (7).

Aug-PDGD (9) is globally exponentially stable, as stated below.

**Theorem 2.** Under Assumption 1 and 2, the Aug-PDGD (9) is globally exponentially stable in the sense that, for any \(\eta > 0, \rho > 0\), there exists constant
\[
\tau_{\text{ineq}} = \frac{\eta \kappa_2^2}{40 \kappa_2 \max(\frac{\rho \kappa_2}{\mu}, \frac{\lambda}{\mu})^2 \max(\frac{\eta}{\tau}, \frac{\eta}{\rho})^2}
\]
and constants \(C_3, C_4 > 0\) which depend on \(\eta, \rho, \kappa_1, \kappa_2, \mu, \ell, \|x(0) - x^*\|, \|\lambda(0) - \lambda^*\|\), s.t. \(\|x(t) - x^*\| \leq C_3 e^{-\frac{\tau}{\tau_{\text{ineq}}}}, \|\lambda(t) - \lambda^*\| \leq C_4 e^{-\frac{\tau}{\tau_{\text{ineq}}}}\).

**Remark 2.** We currently only study the affine inequality constrained case and assume the matrix \(A\) satisfies Assumption 2. We conjecture that the results can be extended in two ways. Firstly, for affine inequality constraints, Assumption 2 can be relaxed to the linear independence constraint qualification, i.e., at the optimizer \(x^*,\) the submatrix of \(A\) associated with the active constraints has full row rank. Secondly, for nonlinear convex constraint \(J(x) \leq 0\) where \(J : \mathbb{R}^n \to \mathbb{R}^m\). Assumption 2 can be replaced by the condition \(\kappa_1 I \preceq \frac{\partial J(x)}{\partial x} (\frac{\partial J(x)}{\partial x})^T \preceq \kappa_2 I\) where \(\frac{\partial J(x)}{\partial x} \in \mathbb{R}^{m \times n}\) is the Jacobian of \(J\) w.r.t. \(x\). We leave these extensions to the future work.

**III. STABILITY ANALYSIS**

In this section, we prove global exponential stability. We also show global exponential stability ensures the geometric convergence rate of the Euler discretization.

**A. The Equality Constrained Case, Proof of Theorem 1**

We stack \(x\) and \(\lambda\) into a larger vector \(z = [x^T, \lambda^T]^T\) and similarly define \(z^* = [x^*^T, (\lambda^*)^T]^T\). We define quadratic Lyapunov function, \(V(z) = (z - z^*)^T P(z - z^*)\) with \(P > 0\) defined by
\[
P = \begin{bmatrix}
\eta c I & \eta A^T \\
\eta A & c I
\end{bmatrix} \in \mathbb{R}^{(m+n) \times (m+n)}
\] (10)
where \(c = 4 \max(\ell, \eta \kappa_2)\).\(^3\) If we can show the following property of \(V(z)\) along the trajectory of the dynamics,
\[
\frac{d}{dt} V(z) \leq -\tau V(z)
\] (11)
for \(\tau = \frac{\eta \kappa_1}{\epsilon} = \min(\frac{\eta \kappa_1}{\epsilon}, \frac{\eta \kappa_1}{\rho \kappa_2})\), then we have proved Theorem 1. The rest of the section will be devoted to proving (11). We start with the following auxiliary Lemma, which can be proved by using mean value theorem. A similar lemma can be found in [8, Lem. A.1], and for completeness we include a proof in our online report [31, Appendix-D].

\(^3\)We have \(P > 0\) as long as \(c^2 > \eta \kappa_2\), which is satisfied by our choice of \(c\).

**Lemma 1.** Under Assumption 1, for any \(x \in \mathbb{R}^n\), there exists a symmetric matrix \(B(x)\) that depends on \(x\), satisfying \(\mu I \preceq B(x) \preceq I\), s.t. \(\nabla f(x) - \nabla f(x^*) = B(x)(x - x^*)\).

With Lemma 1, we can rewrite PDGD (6) as,
\[
\frac{d}{dt} x = \begin{bmatrix}
-\nabla_x L(x, \lambda) - \nabla x^* (x - x^*) \\
\eta \nabla \lambda (x, \lambda) - \eta \nabla \lambda^* (\lambda - \lambda^*)
\end{bmatrix}
\] (12)
\[
= -B(x)(x - x^*) - A^T (\lambda - \lambda^*)
\] (13)
\[
= -B(x) - A^T \frac{\eta A}{\rho} (z - z^*)
\] (14)
\[
= \left( z - z^* \right)^T \left( G(z) + PG(z) \right) (z - z^*)
\] (15)

**Lemma 2.** For any \(z \in \mathbb{R}^{n+m}\), we have
\[
G(z)^T P + PG(z) \leq -\tau P
\] (16)

Lemma 2 and (13) lead to (11), concluding the proof.

**B. The Inequality Constrained Case, Proof of Theorem 2**

We start by emphasizing the notations in this section is independent from the equality constrained case in Section III-A. We stack \(x, \lambda\) into a larger vector \(z = [x^T, \lambda^T]^T\) and similarly define \(z^* = [x^*^T, (\lambda^*)^T]^T\). Next, we define the following quadratic Lyapunov function \(V(z) = (z - z^*)^T P(z - z^*)\) with \(P > 0\) defined by,
\[
P = \begin{bmatrix}
\eta c I & \eta A^T \\
\eta A & c I
\end{bmatrix} \in \mathbb{R}^{(m+n) \times (m+n)}
\] (14)
where \(c = 20 \ell \max(\frac{\rho \kappa_2}{\mu}, \frac{\lambda}{\mu})^2 \max(\frac{\eta}{\tau}, \frac{\eta}{\rho})^2 \frac{\epsilon \kappa_1}{\kappa_2} 4\). Then, the results of Theorem 2 directly follows from the following property of \(V(z)\),
\[
\frac{d}{dt} V(z) \leq -\tau V(z)
\] (15)
where \(\tau = \frac{\eta \kappa_1}{\epsilon} = \frac{\eta \kappa_1}{\eta \kappa_2} 4\). The rest of the section will be devoted to proving (15). To prove (15), we write the Aug-PDGD (9) in a “linear” form. In addition to Lemma 1 we need the following Lemma.

**Lemma 3.** For any \(y, y^* \in \mathbb{R}\), there exists some \(\gamma \in [0, 1]\) that depends on \(y, y^*\) s.t.
\[
\nabla_x H_{\rho} (a_j^T (x - b_j, \lambda_j)) - \nabla_x H_{\rho} (a_j^T x^* - b_j, \lambda_j)
\] (17)
\[
= \gamma_j(z)(a_j^T (x - x^*) a_j + \gamma_j(z)(\lambda_j - \lambda_j^*) a_j)
\] (18)
\[
= \gamma_j(z) a_j^T (x - x^*) e_j + \frac{1}{\rho} \epsilon_j (\gamma_j(z) - 1)(\lambda_j - \lambda_j^*) e_j
\] (19)
\[
= \gamma_j(z) a_j^T (x - x^*) e_j + \frac{1}{\rho} \epsilon_j (\gamma_j(z) - 1)(\lambda_j - \lambda_j^*) e_j
\] (19)
\[
= \gamma_j(z) a_j^T (x - x*) e_j + \frac{1}{\rho} \epsilon_j (\gamma_j(z) - 1)(\lambda_j - \lambda_j^*) e_j
\] (19)

**Proof.** The lemma directly follows from that for any \(y, y^* \in \mathbb{R}\), there exists some \(\gamma \in [0, 1]\), depending on \(y, y^*\) s.t.
\[
\max(y, 0) - \max(y^*, 0) = \gamma \eta(y - y^*)
\] (20)
To see this, when \(y \neq y^*\), set \(\gamma = \max(y, 0) - \max(y^*, 0) / y - y^*\); otherwise, set \(\gamma = 0\)\(\square\).

\(^4\)We have \(P > 0\) as long as \(c^2 > \eta \kappa_2\), which is satisfied by our choice of \(c\).
For any \( z \in \mathbb{R}^{n+m} \), we define notation \( \Gamma(z) = \text{diag}(\gamma_1(z), \ldots, \gamma_m(z)) \in \mathbb{R}^{m \times n} \), where \( \gamma_j(z) \) is from Lemma 3. With notation \( \Gamma(z) \), we can then rewrite the Aug-PDGD (9a) as

\[
\dot{x} = -(\nabla_x L(x, \lambda) - \nabla_x L(x^*, \lambda^*)) = -(\nabla f(x) - \nabla f(x^*))
\]

\[
- \sum_{j=1}^{m} \nabla_x H_\rho(a_j^T x - b_j, \lambda_j) - \nabla_x H_\rho(a_j^T x^* - b_j, \lambda_j^*)
\]

\[
= -B(x)(x - x^*) - \rho A^T \Gamma(z)A(x - x^*) - A^T \Gamma(z)(\lambda - \lambda^*)
\]

where \( \mu I \preceq B(x) \preceq I \) (Lemma 1). We then rewrite (9b),

\[
\lambda = \eta \nabla x L(x, \lambda) - \eta \nabla x L(x^*, \lambda^*)
\]

\[
= \eta \sum_{j=1}^{m} \nabla_x H_\rho(a_j^T x - b_j, \lambda_j) - \nabla_x H_\rho(a_j^T x^* - b_j, \lambda_j^*)
\]

\[
= \eta \Gamma(z)A(x - x^*) + \frac{\eta}{\rho} (\Gamma(z) - I)(\lambda - \lambda^*)
\]

Then, the Aug-PDGD (9) can be written as,

\[
\dot{z} = \left[ -B(x) - \rho A^T \Gamma(z)A \right] \eta \Gamma(z)A \left[ \frac{\eta}{\rho} (\Gamma(z) - I) \right] (z - z^*)
\]

\[
:= G(z)(z - z^*)
\]

Then, \( \frac{d}{dt} V(z) \) can be written as

\[
\frac{d}{dt} V(z) = z^T P(z - z^*) + (z - z^*)^T P \dot{z}
\]

\[
= (z - z^*)^T (G(z)^T P + PG(z))(z - z^*)
\]

Therefore, to prove (15), it is sufficient to show the following Lemma, whose proof is in Appendix-B.

**Lemma 4.** For any \( z \in \mathbb{R}^{n+m} \), we have

\[
G(z)^T P + PG(z) \preceq -\tau P
\]

Lemma 4 and (17) lead to (15), concluding the proof.

**C. Discrete Time Primal-Dual Gradient Algorithm**

Lastly, we briefly discuss the stability of the discretization of (Aug-)PDGD. It is known that the Euler discretization of an exponentially stable dynamical system possesses geometric convergence speed [23], [24], provided the discretization step size is small enough. For completeness, we provide the following Lemma 5, whose proof can be found in our online report [31, Appendix-F].

**Lemma 5.** Consider a continuous-time dynamical system

\[ z = F(z) \] where \( F \) is \( \nu \)-Lipschitz continuous. Suppose \( z^* \) is an equilibrium point and there exists positive definite matrix \( P \), constant \( \tau > 0 \), and Lyapunov function \( V(z) = z^T P \dot{z} \) such that \( \frac{d}{dt} V(z) \leq -\tau V(z) \), where \( \dot{z} := z - z^* \). Then its Euler discretization with step size \( \delta > 0 \),

\[
y(k + 1) = y(k) + \delta F(y(k))
\]

satisfies \( \|y(k) - z^*\| \leq C(e^{-\frac{\delta}{\tau}} + \frac{\kappa P \delta^2}{2}) \), where \( \kappa P \) is the condition number of matrix \( P \), and \( C > 0 \) is some constant that depends on \( \nu, \tau, P \) and \( \|y(0) - z^*\| \). Further, \( e^{-\frac{\delta}{\tau}} + \frac{\kappa P \delta^2}{2} < 1 \) for small enough \( \delta \).

Based on the proof in Section III-A and III-B, both PDGD (6) and Aug-PDGD (9) satisfy the conditions in Lemma 5.

**IV. ILLUSTRATIVE EXAMPLES**

**A. Equality Constrained Case**

We numerically study PDGD with affine equality constraints and quadratic cost functions. We let \( n = 5, m = 2, f(x) = \frac{1}{2}x^T W x \), where \( W = 10I + W_0W_0^T \), and \( W_0 \) is a \( n \)-by-\( n \) Gaussian random matrix. \( A \) and \( b \) are also Gaussian random matrices (or vectors). Since the cost is quadratic, the PDGD (6) becomes an Linear Time-Invariant (LTI) system and we can determine the PDGD decaying rate by numerically calculating the eigenvalues of the resulting LTI system. We plot the decaying rate as a function of \( \eta \) in the upper plot of Fig. 1. We also simulate the PDGD for a selected number of \( \eta \)'s, and plot the distance to equilibrium point as a function of time in the lower plot of Fig. 1. In both plots, we observe that increasing \( \eta \) beyond a certain threshold does not lead to faster decaying rate, an interesting phenomenon that may be worth further studying.

**B. Inequality Constrained Case**

We numerically run the Aug-PDGD on a problem of size \( n = 50, m = 40 \). We use the loss for logistic regression [32] (with synthetic data) as our cost function \( f \). For the affine inequality constraint \( Ax \leq b \), every entry of \( A, b \) is generated independently from standard normal distribution. We fix \( \rho = 1 \) but try different \( \eta \)'s, and show the results in Fig. 2. Similar to the equality constrained case, here we observe that when \( \eta \) is large, the decaying rate doesn’t increase with \( \eta \).

**V. CONCLUSIONS**

In this paper, we study the primal-dual gradient dynamics for optimization with strongly convex and smooth objective and affine equality or inequality constraints. We prove the global exponential stability of PDGD. We also give explicit bounds on the decaying rates. Future work include 1) providing tighter bounds on the decaying rates, especially for...
the inequality constrained case; 2) relaxing Assumption 2 for the inequality constrained case.

REFERENCES


A. Proof of Lemma 2

Recall the definition of $G(z)$ in (12).

$$G(z) = \begin{bmatrix} -B(x) - AT & \eta A \end{bmatrix}$$

It can be seen that $G(z)$ depends on $z$ through $B(x)$, and $B(x)$ satisfies $\mu I \succeq B(x) \preceq \ell I$. In the remaining of this section, we will drop the dependence of $G(z)$ and $B(x)$ on $z$ and $x$, and prove $G(z)^T P + PG \preceq -\tau P$, for any symmetric $B(x)$ satisfying $\mu I \succeq B(x) \preceq \ell I$. Let $Q = -G(z)^T P - PG$, then,

$$Q = -\tau P = \begin{bmatrix} 2\eta c B - 2\eta^2 A^T A - \eta^2 \kappa_1 I & \eta \kappa_1 I \\ \eta \kappa_1 I & 2\eta A^T - \frac{\eta^2 \kappa_1}{c} A^T \end{bmatrix}$$

where we have used $A^T A \succeq \kappa_1 I$, $A^T A \preceq \kappa_2 I$. We will next use the Schur complement argument. Consider

$$\eta \kappa_1 I \succeq \kappa_2 I,$$

where we have used $A^T (A^T - A \eta A^T) \preceq I$. Recall that $c = 4 \max\{\ell, \eta c\}$, $\kappa_1 \leq \kappa_2$ and $\mu \leq \ell$. Then we have, i) $\frac{1}{\eta} \mu c B \succeq \frac{1}{\eta} \mu c I \succeq 2\eta^2 \kappa_2 I \succeq 2\eta^2 \kappa_2 I + \eta \kappa_1 I$, ii) $\frac{1}{\eta} \mu c B \succeq \eta \kappa_1 I$, iii) $\frac{1}{\eta} \mu c B \succeq \eta \kappa_1 I$. Summing them up, we have

$$2\eta B \succeq 2\eta^2 \kappa_2 I + \eta \kappa_1 I + \eta \kappa_1 I + \frac{\eta^2}{c} 2\ell \kappa_1 I + \frac{\eta^2 \kappa_1}{c^2} I.$$
After straightforward calculations, we have

\[ Q \preceq G \quad \text{for any symmetric } B \text{ satisfying } B(I) = \eta B(I) \leq I. \]

We will first lower bound \( Q \) and for any diagonal \( \Gamma = \eta \Gamma(I) \) on \( Z \), and prove \( G \preceq B \leq \Gamma \) for any symmetric \( B \) satisfying \( B(I) \leq \Gamma(I) \) and for any diagonal \( \Gamma \) with each entry bounded in \([0,1]\). Let \( Q = -G^T P - PG \), and

\[ Q = \eta B(I) \quad \text{as long as } c \text{ is sufficiently large. We can verify that our selection of } c \text{ is large enough s.t. } Q_1 - Q_3 Q_2^{-1} Q_3^T \geq 0. \]

Using the Schur complement argument, to prove \( \hat{Q} \geq 0 \) it suffices to prove \( Q_2 \geq 0 \). Then, we will lower bound \( Q_2 \), then upper bound \( Q_3 Q_2^{-1} Q_3^T \), next lower bound \( Q_1 \) and finally show \( Q_1 - Q_3 Q_2^{-1} Q_3^T \geq 0 \).

**Lower bounding** \( Q_2 \). We will use the following lemma, whose proof is deferred to Appendix-C.

**Lemma 6.** If \( c \geq \kappa_2 I \), as long as \( \Gamma \) is a diagonal matrix with each entry bounded in \([0,1]\), we have \( \eta (\Gamma AA^T + AA^T \Gamma) + \eta c(I - \Gamma) \geq \frac{3}{2} \eta AA^T \).

Using Lemma 6, we have \( Q_2 \geq \frac{3}{2} \eta AA^T - \frac{\eta c(I - \Gamma)}{2} \geq \eta AA^T. \)

**Upper bounding** \( Q_3 Q_2^{-1} Q_3^T \). Using the lower bound on \( Q_2 \), we calculate

\[ Q_3 Q_2^{-1} Q_3^T \preceq \frac{1}{\eta} Q_3(\eta A^T A)^{-1} Q_3^T \]

where we intentionally write the last quantity as a function of \( h_1(c) \) depending on \( c \) for reasons to be clear later. We further bound

\[ \eta \|Q_6(\eta A^T A)^{-1} Q_3^T\| \leq \eta \|Q_6\| \|\eta A^T A\|^{-1} \leq \frac{\eta^3}{\rho^2 \kappa_1} \hat{h}_1(c) \]

with \( \eta \) being a function in \( \rho \) and \( \kappa_1 \) is a strictly decreasing function in \( c \), we have \( Q_1 - Q_3 Q_2^{-1} Q_3^T \geq 0 \) as long as \( c \) is sufficiently large. We can verify that our selection of \( c \) is large enough s.t. \( Q_1 - Q_3 Q_2^{-1} Q_3^T \geq 0 \). Due to space limit, we omit the details. Therefore, \( Q \preceq \Gamma \).

**C. Proof of Lemma 6**

Recall that \( \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_m) \) with each \( \gamma_i \in [0,1] \). The matrix of interest is \( M(\gamma_1, \ldots, \gamma_m) = \eta (\Gamma AA^T + AA^T \Gamma) + \frac{2\eta}{\rho}(I - \Gamma) \). It is easy to check \( M(\gamma_1, \ldots, \gamma_m) \) is a convex combination of \( 2^m \) matrices \( M(b_1, \ldots, b_m) : b_i = 0 \text{ or } 1 \). Therefore, to prove the lower bound for \( M(\gamma_1, \ldots, \gamma_m) \), without loss of generality, we only have to prove that for \( k = 0, \ldots, m \)

\[ M_k = M(b_1, \ldots, 1, 0, \ldots, 0) \geq \frac{3}{2} \eta AA^T \]

Notice that \( M_0 = \frac{2mc}{\rho} I \geq \frac{3}{2} \eta AA^T \). \( M_m = 2\eta AA^T \), so (23) is true for \( k = 0 \) and \( m \). Now assume \( 0 < k < m \). We write \( AA^T \) in block matrix form

\[ AA^T = \begin{bmatrix} A_1 & A_3 \\ \lambda_2 & A_2 \end{bmatrix} \]

with \( A_1 \in \mathbb{R}^{k \times k}, A_2 \in \mathbb{R}^{(m-k) \times (m-k)}, A_3 \in \mathbb{R}^{(k) \times (m-k)} \). Then, we can write \( M_k \) as

\[ M_k = \begin{bmatrix} 2\eta \lambda_1 & \frac{\eta \kappa_1}{2} \\ \frac{\eta \kappa_3}{2} \rho \end{bmatrix} \preceq \begin{bmatrix} 2\eta \lambda_1 & \eta \lambda_3 \\ 2\eta \lambda_3 \end{bmatrix} \geq \frac{3}{2} \eta AA^T \]

where we have used the fact that \( A_2 \preceq \lambda_2 \|I\| \leq \kappa_2 \hat{I} \leq \frac{\hat{I}}{\rho} \) (using \( c \geq \rho \)).