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Voltage Control Using Limited Communication ⋆

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Abstract:
In electricity distribution networks, the increasing penetration of renewable energy generation necessitates faster and more sophisticated voltage controls. Unfortunately, recent research shows that local voltage control fails in achieving the desired regulation, unless there is some communication between the controllers. However, the communication infrastructure for distribution systems are less reliable and less ubiquitous as compared to that for the bulk transmission system. In this paper, we design distributed voltage control that use limited communication. That is, only neighboring buses need to communicate few bits between each other before each control step. We investigate how these controllers can achieve the desired asymptotic behavior of the voltage regulation and we provide upper bounds on the number of bits that are needed to ensure a predefined accuracy of the regulation. Finally, we illustrate the results by numerical simulations.

Keywords: Distributed Optimization, Smart Grid, Voltage Regulation, Reactive Power, Limited Communication

1. INTRODUCTION

There is an increasing penetration of distributed energy resources such as renewable energy in distribution networks. Unfortunately, such a penetration causes faster voltage fluctuations than today's distribution networks can handle, see Carvalho et al. (2008). Therefore, to avoid overloading the distribution networks, the integration of renewable energy resources must be accompanied by faster and more sophisticated voltage regulation.

These challenges have motivated recent research on voltage control, where fast voltage fluctuations are regulated through real-time reactive power injections to ensure that the voltage is maintained within an acceptable range. Such fast voltage control can be implemented in the emerging power devices such as inverters. The research efforts have focused on two main directions: local and distributed control strategies. In the local voltage control, control devices at each bus update the reactive power injections using only locally available information, such as local voltage measurements, see Farivar et al. (2013); Li et al. (2014); Zhu and Liu (2016) and references therein. On the other hand, in distributed voltage control schemes, control devices at each bus determine the reactive power injection with additional information communicated from its neighboring buses in the distribution network, see Zhang et al. (2015); Šulc et al. (2014); Bolognani and Zampieri (2013); Bolognani et al. (2015). Local control strategies have the obvious advantage over distributed ones in that they do not rely on communication. However, even though local control strategies perform well in some cases, they may fail to ensure that the voltage is maintained within the accepted range under certain scenarios, as proved by the impossibility result in Cavraro et al. (2016). Therefore, some communication among the local controllers is always needed to guarantee the performance of voltage regulation.

However, the communication capabilities of today's distribution networks generally suffer from very low data rates, Yan et al. (2013); Galli et al. (2011). To compensate for this deficiency, power system operators and industries are currently investing heavily in integrating the distribution networks with a sophisticated communication infrastructure. However, even with the promising capabilities of the future low latency networks, fast real-time control applications, like voltage control, rely on short packages that carry coarsely quantized information, Durisi et al. (2016). Therefore, it is important to develop voltage control with very limited communication for early integration of renewable resources using today's grid limited communication capabilities and also for sustainable developments of the future smart grid.

In this paper, we study a distributed voltage control where only few bits of communication between neighboring buses are needed. In particular, the voltage control device on each bus determines the reactive power injection based on its local voltage measurement and current reactive power injection, in addition to a few bits of information communicated from its physical neighbors. We show that the algorithm can regulate the voltages to an acceptable range, for any predefined accuracy, in a finite number

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of iterations. We also provide an upper bound on the number of communicated bits that are needed to ensure a predefined accuracy of the desired voltage level. Moreover, we prove that this control strategy asymptotically achieves the desired regulation by varying the parameters of the controller with time. Lastly, we also illustrate the results by numerical simulations.

An extended version is found in Magnusson et al. (2017).

1.1 Notation

Vectors and matrices are represented by boldface lower and upper case letters, respectively. The imaginary unit is denoted by $\boldsymbol{i}$, i.e., $\boldsymbol{i} = \sqrt{-1}$. The set of real, complex, and natural numbers are denoted by $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$, respectively. The set of real n vectors and $n{\times}m$ matrices are denoted by $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively. Otherwise, we use calligraphy letters to represent represent sets. We let $\mathcal{S}^{n-1}{=}\{\boldsymbol{x}{\in}\mathbb{R}^n \big| 1{=}||\boldsymbol{x}||\}$ denote the unit sphere. The superscript $(\cdot)^{\mathrm{T}}$ stands for transpose. $\mathrm{diag}(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_n)$ denotes the diagonal block matrix with $\boldsymbol{A}_1,\ldots,\boldsymbol{A}_n$ on the diagonal. $||\cdot||$ denotes the 2-norm.

2. SYSTEM MODEL AND PROBLEM FORMULATION

2.1 System Model: Linearized Power Distribution Network

Consider a radial power distribution network with $N+1$ buses represented by the set $\mathcal{N}_0 = \{0\}\cup\mathcal{N}$, where $\mathcal{N} = \{1,\ldots,N\}$. Bus 0 is a feeder bus and the buses in $\mathcal{N}$ are branch buses. Let $\mathcal{E} \subseteq \mathcal{N}_0\times\mathcal{N}_0$ denote the set of directed flow lines, so if $(i,j)\in\mathcal{E}$ then i is the parent of j. For each i, let $s_i = p_i + \boldsymbol{i}q_i \in \mathbb{C}$, $V_i\in\mathbb{C}$, and $v_i\in\mathbb{R}_+$ denote the complex power injection, complex voltage, and squared voltage magnitude, respectively, at Bus i. For each $(i.j)\in\mathcal{E}$, let $S_{ij} = P_{ij} + \boldsymbol{i}Q_{ij}\in\mathbb{C}$ and $z_{ij} = r_{ij} + \boldsymbol{i}x_{ij}\in\mathbb{C}$ denote the complex power flow and impedance in the line from Bus i to Bus j. To model the relationship between the variables, we use the linearized branch flow model from Baran and Wu (1989), which gives a good approximation in radial distribution networks. [1]

$$-p_i = P_{\sigma_i i} - \sum_{k:(i,k)\in\mathcal{E}} P_{ik},\quad i\in\mathcal{N}, \tag{1a}$$

$$-q_i = Q_{\sigma_i i} - \sum_{k:(i,k)\in\mathcal{E}} Q_{ik},\quad i\in\mathcal{N}, \tag{1b}$$

$$v_j - v_i = -2r_{ij}P_{ij} - 2x_{ij}Q_{ij},\quad (i,j)\in\mathcal{N}, \tag{1c}$$

where σ_i is the parent of bus $i\in\mathcal{N}$, i.e., the unique $\sigma_i\in\mathcal{N}_0$ with $(\sigma_i,i)\in\mathcal{E}$. By rearranging Equation (1) we get that

$$\boldsymbol{v} = \boldsymbol{A}\boldsymbol{q} + \boldsymbol{B}\boldsymbol{p} + \mathbf{1}v_0, \tag{2}$$

where $\boldsymbol{v} = [v_1,\ldots,v_N]^{\mathrm{T}}$, $\boldsymbol{q} = [q_1,\ldots,q_N]^{\mathrm{T}}$, $\boldsymbol{p} = [p_1,\ldots,p_N]^{\mathrm{T}}$,

$$\boldsymbol{A}_{ij} = 2\sum_{(h,k)\in\mathcal{P}_i\cap\mathcal{P}_j} x_{hk},\quad \text{and}\quad \boldsymbol{B}_{ij} = 2\sum_{(h,k)\in\mathcal{P}_i\cap\mathcal{P}_j} r_{hk},$$

where $\mathcal{P}_i\subseteq\mathcal{E}$ is the set of edges in the path from Bus 0 to Bus i. We use the following result in the algorithm development.

[1] The results also directly apply to the linearized power flow model in Cavraro et al. (2016).

Lemma 1. $\boldsymbol{A}$ is a positive definite matrix whose inverse has the following structure

$$a_{ij}{:=}[\boldsymbol{A}^{-1}]_{ij}{=}\begin{cases}\dfrac{1}{2}\left(x^{-1}_{\sigma_i i}{+}\displaystyle\sum_{k:(i,k)\in\mathcal{E}} x^{-1}_{ik}\right) & \text{if } i{=}j,\\ -\dfrac{1}{2}x^{-1}_{ij} & \text{if } (i,j){\in}\mathcal{E}\\ & \text{or } (j,i){\in}\mathcal{E},\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

Proof. It is proved in (Farivar et al., 2013, Lemma 1) that $\boldsymbol{A}$ is positive definite. Direct calculations show that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$. □

We now introduce the Voltage Regulation Problem.

2.2 Voltage Regulation Problem

Suppose that the real power injection $\boldsymbol{p}$ at each bus has been decided. We also write the reactive power injection $\boldsymbol{q}$ as two parts, i.e., $\boldsymbol{q} = \boldsymbol{q}^{\mathrm{V}} + \boldsymbol{q}^{\mathrm{U}}$, where $\boldsymbol{q}^{\mathrm{V}}$ is the adjustable reactive power that can be used for voltage regulation and $\boldsymbol{q}^{\mathrm{U}}$ denotes other reactive power injection that cannot be changed by the voltage control devices. Then the goal of the voltage regulation problem is to find feasible voltages $\boldsymbol{v}$ and reactive powers $\boldsymbol{q}^{\mathrm{V}}$ so that the physical relationship (2) holds and that $\boldsymbol{v}$ and $\boldsymbol{q}^{\mathrm{V}}$ are inside some feasible operation range $[\boldsymbol{v}^{\min},\boldsymbol{v}^{\max}]$ and $[\boldsymbol{q}^{\min},\boldsymbol{q}^{\max}]$. Formally, the voltage regulation problem is to find the reactive power injection $\boldsymbol{q}^{\mathrm{V}}$ so that,

$$\boldsymbol{v}(\boldsymbol{q}^{\mathrm{V}}) = \boldsymbol{A}\boldsymbol{q}^{\mathrm{V}} + \boldsymbol{d}, \tag{4a}$$

$$\boldsymbol{v}^{\min} \le \boldsymbol{v}(\boldsymbol{q}^{\mathrm{V}}) \le \boldsymbol{v}^{\max} \tag{4b}$$

$$\boldsymbol{q}^{\min} \le \boldsymbol{q}^{\mathrm{V}} \le \boldsymbol{q}^{\max} \tag{4c}$$

where $\boldsymbol{d} = \boldsymbol{A}\boldsymbol{q}^{\mathrm{U}} + \boldsymbol{B}\boldsymbol{p} + \mathbf{1}v_0$. In the rest of the paper we drop the superscript V from $\boldsymbol{q}^{\mathrm{V}}$ for sake of notational ease, without causing any confusion. We also assume, without loss of generality, that every bus in $\mathcal{N}$ can adjust its reactive power.

In this paper, we study distributed control laws for finding feasible reactive power injections and voltages that satisfy Equation (4). In particular, each bus updates its reactive power injection by following a local control law that depends only on information available at the bus and limited information communicated from neighboring buses. Formally, each bus $i\in\mathcal{N}$ updates its reactive power injection according to the following rule

$$\boldsymbol{q}_i(t+1) = K_i(\text{Local\_Information}_i(t), \bar{\boldsymbol{b}}_i(t)),$$

where t is the iteration index and K_i is the local control law at bus i. The function K_i depends on the local information, which we denote by $\text{Local\_Information}_i(t)$, at Bus i at iteration t. Formally,

$$\text{Local\_Information}_i(t) = (\boldsymbol{q}_i(0),\ldots,\boldsymbol{q}_i(t),\boldsymbol{v}_i(0),\ldots,\boldsymbol{v}_i(t)),$$

and $\bar{\boldsymbol{b}}_i(t)$, the information available from neighboring buses of Bus i at time t is given by

$$\bar{\boldsymbol{b}}_i(t) = ((\boldsymbol{b}_j(0))_{j\in\mathcal{N}_i},\ldots,(\boldsymbol{b}_j(t))_{j\in\mathcal{N}_i}),$$

where $\mathcal{N}_i = \{j\in\mathcal{N} | (i,j)\in\mathcal{E} \text{ or } (i,j)\in\mathcal{E}\}$ and $\boldsymbol{b}_j(t)$ is the information that Bus j communicates to its neighbors at iteration t. In Section 3 we provide the explicit control algorithm, where $\boldsymbol{b}_j(t)$ is communicated using 2 bits per iteration. Before that, first we need to provide some related background in the following subsection.

2.3 Distributed Algorithm Based on Lagrangian Duality

We now review distributed algorithms for the Voltage Regulation Problem based on dual decomposition Bolognani and Zampieri (2013); Bolognani et al. (2015); Cavraro et al. (2016). In particular, we find the feasible point to the Voltage Regulation Problem that solves the following optimization problem

$$\begin{aligned} \underset{\boldsymbol{q}}{\text{minimize}} \quad & \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{q} \\ \text{subject to} \quad & \boldsymbol{v}^{\min} \leq \boldsymbol{v}(\boldsymbol{q}) \leq \boldsymbol{v}^{\max} \\ & \boldsymbol{q}^{\min} \leq \boldsymbol{q} \leq \boldsymbol{q}^{\max}. \end{aligned} \tag{5}$$

Problem (5) is convex because of Lemma 1. We obtain a distributed algorithm for solving Optimization Problem (5) by considering its dual problem. The Dual Problem is given by

$$\begin{aligned} \underset{\boldsymbol{\lambda},\boldsymbol{\mu}}{\text{maximize}} \quad & D(\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \text{subject to} \quad & \boldsymbol{\lambda},\boldsymbol{\mu} \geq 0, \end{aligned} \tag{6}$$

where $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{\min}, \boldsymbol{\lambda}^{\max})$ and $\boldsymbol{\mu} = (\boldsymbol{\mu}^{\min}, \boldsymbol{\mu}^{\max})$ are the dual variables and $D : \mathbb{R}^{4N} \to \mathbb{R}$ is the dual function, see Chapter 5 in Bertsekas (1999) for the details. The dual gradient is

$$\nabla D(\boldsymbol{\lambda},\boldsymbol{\mu}) = \begin{bmatrix} \nabla^{\boldsymbol{\lambda}} D(\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \nabla^{\boldsymbol{\mu}} D(\boldsymbol{\lambda},\boldsymbol{\mu}) \end{bmatrix} \tag{7}$$

where

$$\nabla^{\boldsymbol{\lambda}} D(\boldsymbol{\lambda},\boldsymbol{\mu}) = \begin{bmatrix} \boldsymbol{v}^{\min} - \boldsymbol{v}(\boldsymbol{q}(\boldsymbol{\lambda},\boldsymbol{\mu})) \\ \boldsymbol{v}(\boldsymbol{q}(\boldsymbol{\lambda},\boldsymbol{\mu})) - \boldsymbol{v}^{\max} \end{bmatrix},$$

$$\nabla^{\boldsymbol{\mu}} D(\boldsymbol{\lambda},\boldsymbol{\mu}) = \begin{bmatrix} \boldsymbol{q}^{\min} - \boldsymbol{q}(\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \boldsymbol{q}(\boldsymbol{\lambda},\boldsymbol{\mu}) - \boldsymbol{q}^{\max} \end{bmatrix},$$

and

$$\boldsymbol{q}(\boldsymbol{\lambda},\boldsymbol{\mu}) = \boldsymbol{\lambda}^{\min} - \boldsymbol{\lambda}^{\max} + \boldsymbol{A}^{-1}\boldsymbol{\mu}^{\min} - \boldsymbol{A}^{-1}\boldsymbol{\mu}^{\max}. \tag{8}$$

It can be checked that the dual function is quadratic so the dual gradient $\nabla D(\cdot)$ is L-Lipschitz continuous. Therefore, the gradient decent method

$$\boldsymbol{\lambda}(t+1) = \lceil \boldsymbol{\lambda}(t) + \gamma \nabla^{\boldsymbol{\lambda}} D(\boldsymbol{\lambda}(t),\boldsymbol{\mu}(t)) \rceil_{+} \tag{9a}$$

$$\boldsymbol{\mu}(t+1) = \lceil \boldsymbol{\mu}(t) + \gamma \nabla^{\boldsymbol{\mu}} D(\boldsymbol{\lambda}(t),\boldsymbol{\mu}(t)) \rceil_{+} \tag{9b}$$

converges to the set of optimal dual variables for appropriate step-size γ (Nesterov, 2004, Chapter 2). These steps can be carried out in a distributed fashion as follows.

Alg 1: Voltage Control - Infinite Bandwidth

Initialization: Set $t = 0$, $\boldsymbol{\lambda}_i(0) = (\boldsymbol{\lambda}_i^{\min}(0), \boldsymbol{\lambda}_i^{\max}(0)) = (0,0)$ and $\boldsymbol{\mu}_i(0) = (\boldsymbol{\mu}_i^{\min}(0), \boldsymbol{\mu}_i^{\max}(0)) = (0,0)$ for all $i \in \mathcal{N}$.

Local Computation: Each bus $i \in \mathcal{N}$ computes its next reactive power injection $\boldsymbol{q}_i(t+1)$ by solving the local subproblem (8), i.e.,

$$\boldsymbol{q}_i(t+1) = \boldsymbol{\lambda}_i^{\max}(t) - \boldsymbol{\lambda}_i^{\min}(t) + \sum_{j \in \mathcal{N}_i} a_{ij}(\boldsymbol{\mu}_j^{\min}(t) - \boldsymbol{\mu}_j^{\max}(t)).$$

Local Control: Each bus $i \in \mathcal{N}$ injects the reactive power $\boldsymbol{q}_i(t+1)$.

Local Measurement: Each bus $i \in \mathcal{N}$ measures the voltage magnitude $\boldsymbol{v}_i(\boldsymbol{q}(t))$, given by the physical relationship (4a).

Communication: Each bus $i \in \mathcal{N}$ communicates $\boldsymbol{q}_i(t+1) - \boldsymbol{q}_i^{\max}$ and $\boldsymbol{q}_i^{\min} - \boldsymbol{q}_i(t+1)$ to its neighbors $j \in \mathcal{N}_i$

Local Computation: Each bus $i \in \mathcal{N}$ updates its dual variables

$$\boldsymbol{\lambda}_i(t+1) = \left\lceil \boldsymbol{\lambda}_i(t) + \gamma(t) \begin{bmatrix} \boldsymbol{v}_i(\boldsymbol{q}(t+1)) - \boldsymbol{v}_i^{\max} \\ \boldsymbol{v}_i^{\min} - \boldsymbol{v}_i(\boldsymbol{q}(t+1)) \end{bmatrix} \right\rceil_{+}, \tag{10}$$

$$\boldsymbol{\mu}_i(t+1) = \left\lceil \boldsymbol{\mu}_i(t) + \gamma(t) \begin{bmatrix} \boldsymbol{q}(t+1)) - \boldsymbol{q}_i^{\max} \\ \boldsymbol{q}_i^{\min} - \boldsymbol{q}(t+1) \end{bmatrix} \right\rceil_{+}. \tag{11}$$

Bus i also updates a local copy of $\boldsymbol{\mu}_j(t+1)$ for each neighbor $j \in \mathcal{N}_i$ using Equation (11).

Update Iteration Index: $t = t + 1$.

Some of the results in the paper use the standard Slater condition in convex optimization.

Assumption 1. (Slaters Condition). Problem (5) is strictly feasible, i.e., there exists $\bar{\boldsymbol{q}} \in \mathbb{R}^N$ such that $\boldsymbol{v}^{\min} < \boldsymbol{v}(\bar{\boldsymbol{q}}) < \bar{\boldsymbol{v}}^{\max}$ and $\boldsymbol{q}^{\min} < \bar{\boldsymbol{q}} < \boldsymbol{q}^{\max}$.

The Slater condition ensure that the optimal value of the primal problem (5) is the same as the optimal value of the dual Problem (6). Moreover, the Slater condition ensures that the set of optimal dual variables is bounded.

Lemma 2. Suppose the Slater Condition (Assumption 1). Then the set of optimal dual variables $\mathcal{Z}^{\star}$ is bounded.

Proof. Follows directly from Lemma 1 in Nedic and Ozdaglar (2009). □

3. VOLTAGE CONTROL WITH LIMITED COMMUNICATION

3.1 Algorithm

In the voltage control algorithm Alg 1 in Section 2.3 the controllers communicate real numbers, $q_i(t+1) - q_i^{\max}$ and $q_i^{\min} - q_i(t+1)$ to its neighbors. This can be challenging in practice as communication among controllers is generally constrained to low data rates. To compensate for that, we now provide a quantized variant of Alg 1 where controllers only need to communicate few bits to their neighbours. The algorithm can formally be expressed as the following variant of the gradient descent method in Equations (9a) and (9b) changed to be

$$\boldsymbol{\lambda}(t+1) = \lceil \boldsymbol{\lambda}(t) + \alpha(t) \nabla^{\boldsymbol{\lambda}} D(\boldsymbol{\lambda}(t)) \rceil_{+}, \tag{12a}$$

$$\boldsymbol{\mu}(t+1) = \lceil \boldsymbol{\mu}(t) + \beta(t) \mathtt{sign}(\nabla^{\boldsymbol{\mu}} D(\boldsymbol{\mu}(t))) \rceil_{+}, \tag{12b}$$

where $\alpha(t), \beta(t) > 0$ are step-sizes and the primal variables are updated according to (8). The algorithm can be realized as follows:

Alg 2: Voltage Control - Limited Bandwidth

Initialization: Set $t = 0$, $\boldsymbol{\lambda}_i(0) = (\boldsymbol{\lambda}_i^{\min}(0), \boldsymbol{\lambda}_i^{\max}(0)) = (0,0)$ and $\boldsymbol{\mu}_i(0) = (\boldsymbol{\mu}_i^{\min}(0), \boldsymbol{\mu}_i^{\max}(0)) = (0,0)$ for all $i \in \mathcal{N}$.

Local Computation: Each bus $i \in \mathcal{N}$ computes its next reactive power injection $\boldsymbol{q}_i(t+1)$ by solving the local subproblem (8), i.e.,

$$\boldsymbol{q}_i(t{+}1){=}\boldsymbol{\lambda}_i^{\max}(t){-}\boldsymbol{\lambda}_i^{\min}(t){+}\sum_{j\in\mathcal{N}_i} a_{ij}(\boldsymbol{\mu}_j^{\min}(t){-}\boldsymbol{\mu}_j^{\max}(t)).$$

Bus i can then also computes the communicated signal

$$\boldsymbol{b}_i(t) = \mathtt{sign}\begin{bmatrix} \boldsymbol{q}(t{+}1) - \boldsymbol{q}_i^{\max} \\ \boldsymbol{q}_i^{\min} - \boldsymbol{q}(t{+}1) \end{bmatrix}. \tag{13}$$

Local Control: Each bus $i \in \mathcal{N}$ injects the reactive power $\boldsymbol{q}_i(t{+}1)$.

Local Measurement: Each bus $i \in \mathcal{N}$ measures the voltage magnitude $\boldsymbol{v}_i(\boldsymbol{q}(t))$, given by the physical relationship (4a).

Communication: Each bus $i \in \mathcal{N}$ communicates $\boldsymbol{b}_i(t)$ to each of its neighbours $j \in \mathcal{N}_i$ using a two bits.

Local Computation: Each bus $i \in \mathcal{N}$ updates its dual variables

$$\boldsymbol{\lambda}_i(t{+}1){=}\left\lceil \boldsymbol{\lambda}_i(t){+}\alpha(t)\begin{bmatrix} \boldsymbol{v}_i(\boldsymbol{q}(t{+}1)) - \boldsymbol{v}_i^{\max} \\ \boldsymbol{v}_i^{\min} - \boldsymbol{v}_i(\boldsymbol{q}(t{+}1)) \end{bmatrix}\right\rceil_+, \tag{14}$$

$$\boldsymbol{\mu}_i(t{+}1){=}\lceil \boldsymbol{\mu}_i(t){+}\beta(t)\boldsymbol{b}_i(t)\rceil_+. \tag{15}$$

Bus i also updates a local copy of $\boldsymbol{\mu}_j(t{+}1)$ for each neighbor $j \in \mathcal{N}_i$ using Equation (15).

Update Iteration Index: $t = t + 1$.

The local computations, measurements, and control are similar as in the algorithm in Cavraro et al. (2016). However, the communication step is different, since here quantized dual gradient is communicated. In particular, at every iteration t, each bus $i \in \mathcal{N}$ communicates $\boldsymbol{b}_i(t)$, defined in Equation (13), to its neighbours $j \in \mathcal{N}_i$. Since $\boldsymbol{b}_i(t) \in \{-1, 1\}^2$, 2 bits are needed to communicate $\boldsymbol{b}_i(t)$.

3.2 Main Convergence Results

The main results of this paper is to prove that Alg. 2 can converge to a solution to the voltage regulation problem in Section 2.2, with proper choice of step-sizes. In particular, we show that the iterates $\boldsymbol{q}(t)$ and $\boldsymbol{v}(\boldsymbol{q}(t))$ satisfy Equation (4) (i) approximately after finite number of iterations, when the step-sizes are constant, and (ii) asymptotically for some time-varying step-sizes. We measure the feasibility of reactive power injection $\boldsymbol{q} \in \mathbb{R}^N$ as follows

$$\mathtt{dist}(\boldsymbol{q}, \mathcal{Q}) = \min_{\bar{\boldsymbol{q}}\in\mathcal{Q}} ||\boldsymbol{q} - \bar{\boldsymbol{q}}||, \tag{16}$$

where

$$\mathcal{Q} = \left\{ \boldsymbol{q} \in \mathbb{R}^N \middle| \begin{bmatrix} \boldsymbol{v}^{\min} \\ \boldsymbol{q}^{\min} \end{bmatrix} \le \begin{bmatrix} \boldsymbol{v}(\boldsymbol{q}) \\ \boldsymbol{q} \end{bmatrix} \le \begin{bmatrix} \boldsymbol{v}^{\max} \\ \boldsymbol{q}^{\max} \end{bmatrix} \right\}. \tag{17}$$

The main results of this paper can now be formally expressed in the following three theorems.

Theorem 1. Consider the algorithm given in Equations (12a) and (12b). For all $\epsilon > 0$ and step-sizes $\alpha, \beta > 0$ such that

$$\alpha < \min\left\{\frac{2}{L}, 1\right\}, \tag{18}$$

$$\beta < \min\left\{\frac{\epsilon^2}{(2N)^{3/2}\max\{L, 1/N\}}, \sqrt{\frac{\alpha(1 - L\alpha/2)\epsilon^2}{2NL}}\right\}, \tag{19}$$

where L is a lipschitz constant of ∇D. Then there exists $T \in \mathbb{N} \cup \{0\}$ such that

$$\mathtt{dist}(\boldsymbol{q}(T), \mathcal{Q}) \le \epsilon$$

where T is upper bounded by

$$T \le \left\lceil \frac{D^\star - D(\boldsymbol{z}(0))}{\delta(\alpha, \beta)} \right\rceil, \tag{20}$$

where $\delta(\alpha, \beta) > 0$ is defined in Equation (28).

Theorem 1 shows how to choose the step-sizes so that Alg. 2 can solve the voltage regulation problem with ϵ-accuracy, for any $\epsilon > 0$, in finite number of iterations. However, the theorem does not ensure that $\mathtt{dist}(\boldsymbol{q}(t), \mathcal{Q}) \le \epsilon$ for all $t \ge T$. The next result shows that there always exist step-sizes so that $\mathtt{dist}(\boldsymbol{q}(t), \mathcal{Q}) \le \epsilon$ for all $t \ge T$, for some $T \in \mathbb{N}$.

Theorem 2. Suppose that Assumption 1 holds and consider the algorithm given in Equations (12a) and (12b). Then for any $\epsilon > 0$ there exist step-sizes $\alpha, \beta > 0$ and $T \in \mathbb{N}$ such that

$$\mathtt{dist}(\boldsymbol{q}(t), \mathcal{Q}) \le \epsilon, \text{ for all } t \ge T. \tag{21}$$

Finally, the following theorem, shows how to choose the step-sizes to obtain asymptotic convergence.

Theorem 3. Suppose that Assumption 1 holds and consider the algorithm given in Equations (12a) and (12b). If the step-sizes are chosen as

$$\alpha(t) \le \min\left\{\frac{L}{2}, 1\right\}, \quad \lim_{t\to\infty}\beta(t) = 0, \quad \text{and} \quad \sum_{t=0}^{\infty}\beta(t) = \infty$$

then

$$\lim_{t\to\infty} \boldsymbol{q}(t) = \boldsymbol{q}^\star \in \mathcal{Q},$$
$$\lim_{t\to\infty} \mathtt{dist}(\boldsymbol{z}(t), \mathcal{Z}^\star) = 0,$$

where $\boldsymbol{q}^\star$ and $\mathcal{Z}^\star$ are the optimiser of Problem (5) and the set of optimizers of the Dual Problem (6).

We prove these results in the next section.

4. CONVERGENCE ANALYSIS

4.1 Preliminary Results

In this section, we derive some necessary lemmas needed to prove the main convergence results in Section 3.2. We then prove Theorems 1, 2, and 3 in Sections 4.2, 4.3, and 4.4, respectively.

To the prove the convergence, we use the connection between the feasibility measure in Equation (16) and the optimality of the dual iterates $\boldsymbol{z}(t)$ established in the following lemma.

Lemma 3. Consider the function

$$V(\boldsymbol{z}) = ||\boldsymbol{z} - \lceil \boldsymbol{z} + \nabla D(\boldsymbol{z})\rceil_+||. \tag{22}$$

Following holds:

a) A feasible dual variable $\boldsymbol{z} \in \mathbb{R}_+^{4N}$ is an optimal solution to the Dual Problem (6) if and only if $V(\boldsymbol{z}) = 0$.
b) For all $\boldsymbol{z} \in \mathbb{R}_+^{4N}$

$$\mathtt{dist}(\boldsymbol{q}(\boldsymbol{z}), \mathcal{Q}) \le V(\boldsymbol{z}), \tag{23}$$

where $\boldsymbol{q}(\boldsymbol{z})$ is defined in Equation (8).

Proof. See Magnusson et al. (2017). □

The lemma shows that the feasibility measure in Equation (16) is upper bounded $V(\boldsymbol{z})$. This is helpful because, as we show in Section 4.2, we can make $V(\boldsymbol{z})$ arbitrarily small with proper step-size chooses in the quantized dual descent algorithm in Equation (12a) and (12b). We will also use the following technical result in the derivations.

Lemma 4. For all $\beta \in [0,1]$, $z \in \mathbb{R}$ and $x, \alpha_1, \alpha_2 \in \mathbb{R}_+$ with $\alpha_1 \leq |x - \lceil x + \alpha_2 \mathtt{sign}(z) \rceil_+|$ following holds

$$\beta|x - \lceil x + z\rceil^+| \leq |x - \lceil x + \beta z\rceil^+|, \tag{24}$$
$$\alpha_1 = |x - \lceil x + \alpha_1 \mathtt{sign}(z)\rceil_+| \tag{25}$$
$$0 \leq z(\lceil x + \alpha_1 z\rceil^+ - x). \tag{26}$$

Proof. See Magnusson et al. (2017). □

We are now ready to proceed to the proofs of the main results.

4.2 Proof of Theorem 1

We now prove Theorem 1. The main step of the proof is illustrated in the following lemma.

Lemma 5. Suppose $\epsilon > 0$ and $\boldsymbol{z} = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}_+^{4N}$ are such that $V(\boldsymbol{z}) \geq \epsilon$. Choose the step-size $\alpha, \beta > 0$ as in Equations (18) and (19). Then for

$$\bar{\boldsymbol{z}} = (\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}) = \left[\begin{matrix}\boldsymbol{\lambda} \\ \boldsymbol{\mu}\end{matrix} + \begin{matrix}\alpha \nabla^{\boldsymbol{\lambda}} D(\boldsymbol{\lambda}) \\ \beta \mathtt{sign}(\nabla^{\boldsymbol{\mu}} D(\boldsymbol{\mu}))\end{matrix}\right]$$

following holds

$$D(\lceil \bar{\boldsymbol{z}} \rceil^+) \geq D(\boldsymbol{z}) + \delta(\alpha, \beta) \tag{27}$$

where

$$\delta(\alpha,\beta) = \min\left\{\left(\alpha - \frac{L}{2}\alpha^2\right)\frac{\epsilon^2}{2} - NL\beta^2, \left(\frac{\epsilon^2}{(2N)^{3/2}L} - \beta\right)\beta L N\right\}$$
$$> 0 \tag{28}$$

where L is an Lipschitz constant on $\nabla D(\cdot)$.

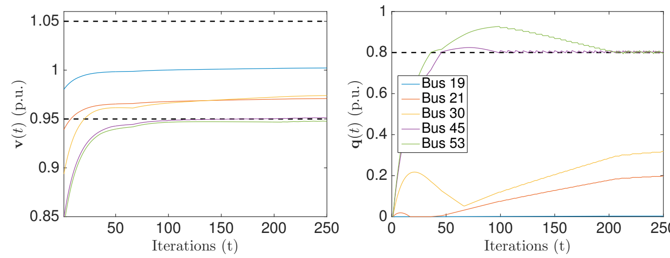
Proof. See Magnusson et al. (2017). □

Note that Lemma 5 is similar to Lemma 4 in Magnússon et al. (2016). However, in Lemma 4 in Magnússon et al. (2016) the gradient is assumed to be bounded. Moreover, unlike in Magnússon et al. (2016) the dual algorithm in this paper is an hybrid between the non-quantizes gradient step in Equation (12a) and the quantized gradient step in (12b). Therefore, the results in Magnússon et al. (2016) do not apply here.

The importance of Lemma 5 is that it shows that for any given $\epsilon > 0$, we can choose step-sizes $\alpha, \beta > 0$ so that if $V(\boldsymbol{z}) > \epsilon$ then the dual objective function value is improved by taking a step of the algorithm in Equations (12a) and (12b). We now use this intuition to prove Theorem 1.

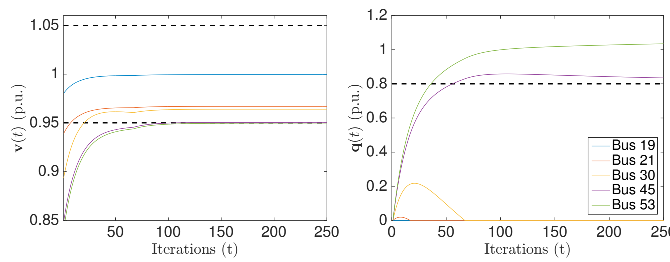
Proof. [of Theorem 1] From Lemma 3 it suffices to prove that $V(\boldsymbol{q}(T)) \leq \epsilon$. We prove the result by contradiction. Suppose $V(\boldsymbol{z}(t)) > \epsilon$ for $t = 0, 1, \ldots, T = \lceil (D^\star - D(\boldsymbol{z}(0)))/\delta(\alpha,\beta) \rceil$. Then by Lemma 5 we have that

$$\begin{aligned} 0 &\leq D^\star - D(\boldsymbol{z}(T)) \\ &\leq D^\star - D(T-1) - \delta(\alpha,\beta) \\ &\leq D^\star - D(0) - T\delta(\alpha,\beta), \end{aligned} \tag{29}$$



(a) Voltage magnituide (b) Reactive power injection

Fig. 1. The studied limited communication control law.



(a) Voltage magnituide (b) Reactive power injection

Fig. 2. The local control law from Li et al. (2014).

where the final inequality comes by the fact that

$$T \leq (D^\star - D(\boldsymbol{z}(0)))/\delta(\alpha,\beta).$$

Then

$$0 \leq D^\star - D(0) - T\delta(\alpha,\beta) < 0,$$

which is clearly a contradiction. □

4.3 Proof of Theorem 2

Due to space limitations we give the proof in the extended version of this paper in Magnusson et al. (2017).

4.4 Proof of Theorem 3

Due to space limitations we give the proof in the extended version of this paper in Magnusson et al. (2017).

5. SIMULATIONS

We illustrate the results on the 56 bus radial distribution network in Farivar et al. (2012). We let Bus 1 be the feeder bus and let buses 19, 21, 30, 45, and 53 have the ability to inject reactive power, e.g., from a inverter of photovoltaic generator. All quantities are given in the per unit (p.u) system.

We use the limited communication algorithm of this paper to regulate the voltages so that they are within $\pm 0.05\%$ range of the the nominal value 1, so $v^{\min} = 0.95$ and $v^{\max} = 1.05$. The feasible operation range of the reactive power injected at buses 19, 21, 30, 45, and 53 is $q^{\min} = 0$ and $q^{\max} = 0.8$. We let the reactive power that cannot be adjusted, denoted by $\boldsymbol{q}^U$ Section 2.2, be 0 at each bus. We let the real power injection $\boldsymbol{p}$ be $3p.u$ at buses 19, 21, 30, 45, and 53 and be uniform random on the interval $[-1, 0]$ for the other buses. The voltage magnitude at the feeder bus is $v_0 = 1$. We use the step-sizes $\alpha = 0.5$ and $\beta = 10^{-4}$.

Fig. 1 illustrates the results. Fig. 1a and Fig. 1b show the voltages and the reactive power injections per iteration, respectively. The results show that after roughly 200 iterations the algorithm converges approximately to a feasible operation point. Therefore, roughly 400 bits of communication per bus are needed to reach that operation point. The reactive power injections overshoot the feasible region at buses 45 and 53 for couple of iterations but then oscillate around the upper bound $q^{\max} = 0.8$ after iteration 200. The voltage magnitude at buses 19, 21, 30, and 45 converge to the feasible region after 172 iterations. The voltage magnitude at bus 53 reaches the lower bound $v^{\max} = 0.95$ with in small accuracy after roughly 200 iterations, escalates around 0.9477. Since a constant step-size is used we can not expect an asymptotic convergence to a feasible point. However, from Theorems 1 and 2, we can reach a solution higher accuracy by choosing smaller step-size β, at the cost of slower convergence.

Fig. 2b illustrate the performance of the local control law from Li et al. (2014) on the same problem data. Note that this local control law is a special case of the algorithm in this paper with $\beta = 0$. The results show that even though the local control law converges to feasible voltage magnitudes it does not converge to the feasible operation range of the reactive power injections. Therefore, adding few bits of communication to the control law in Li et al. (2014), as we have done, enforces convergence to a feasible operation point with respect to the reactive power injections.

6. CONCLUSION

This paper studied distributed voltage control algorithms where only few bits of communication between neighboring buses are needed. The convergence of these algorithms was studied and their practical applicability illustrated in simulations. Future work will study the trade-offs between the data rate and control performance.

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