# Mechanism Design for Reliability in Demand Response with Uncertainty

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*Abstract*— **In this paper, we consider a two-stage demand response (DR) program where a DR aggregator calls upon customers to reduce demands at stage II in order to clear a targeted amount of electricity supply deficit. At stage I, customers only possess distributions of their load shedding costs instead of accurate values because of the intrinsic uncertainty of load shedding. We design an outcome-contingent mechanism in which customers reports their private types of the cost distributions and the DR aggregator select customers based on the reported information. The mechanism guarantees incentive compatibility and individual rationality. In addition, the mechanism ensures a high reliability of the DR. That is, the deviation between the total reduced loads and the targeted amount is small. We provide both theoretical analysis and numerical studies to demonstrate the high reliability.**

## I. Introduction

Customer participation has been playing an important role in transforming the electricity grid into a more energy efficient and sustainable one. Consequently, demand response (DR), which uses incentives to change customer behavior of electricity usage, has attracted lots of discussion [1] [2] [3] [4] [5] [6], and real-world applications [7] [8] [9]. For instance, voluntary day-ahead DR is widely used in practice, e.g., the DR programs by PJM [9] and PGE [7]. In this setup, a DR aggregator announces an electricity deficit to customers and schedules a DR plan in the day ahead of DR. Customers join DR plan voluntarily for monetary credits (rewards) and are not obligated to do DR when the deficit happens, even if they have reported that they would respond.

The uncertainty of whether customers will commit to perform DR or not causes a serious reliability problem. That is, if there is a targeted amount of electricity deficit to clear, it is difficult to ensure a small deviation between the true total reduced demands and the targeted value. A large deviation will further cause difficulties in maintaining the supply-demand balance for the electricity grid [10], [11]. One way to increase customers' commitment may be through exerting a penalty of un-commitment on those who have reported that they would respond. However, a simple high penalty will fail because it may discourage customers from participating in the DR at the very beginning. But if the amount of penalty is carefully chosen, this might be avoided. The difficulty lies in how to choose the right penalty, especially due to the fact that the DR aggregator lacks private information from customers. This paper seeks a mechanism design (MD) approach for the DR aggregator to acquire information from customers and then choose the right customers and set the right amount of penalties in order to achieve high reliability, i.e., a small deviation.

Mechanism design (MD) has been used in many DR research projects [12] [13] [14] [15]. In MD, customers report their private information, such as cost of load shedding, to a DR aggregator; the DR aggregator selects DR customers and determines payment rules for the DR program. However, most of those day-ahead DR mechanisms neglect one important aspect of this problem, i.e., the uncertainty in customers' costs of load shedding [13] [15]. The uncertainty originates from collecting cost information for a future DR event. Customers, especially small ones, are usually not fully certain about their future costs of load shedding due to, for example, weather conditions or emergencies. Neglecting the cost uncertainty inevitably triggers strategic reporting, which results in defective control and estimation on customers' behaviors and thereby poor reliability. [14] is a previous attempt to deal with cost uncertainties. However, it only considers a specific case when customers costs follow binomial distribution.

To deal with the problems above, we design a mechanism for a two-stage DR program which involves customers with uncertain random costs of load shedding. The goal is to call upon customers to reduce demands in order to clear a targeted amount of electricity supply deficit. We assume that the targeted amount is $M$ units and each customer is either able to shed one unit of load or not. At stage I, customers only possess distributions of their load shedding costs. The mechanism collects reports from customers for their cost distribution types. Based on the reported information, the mechanism chooses a subset of customers whom are asked to shed loads at stage II. Those customers receive a fixed reward but are also subject to a potential penalty if they do not perform DR at stage II. The penalties are determined by the mechanism as well. The selection rules and the penalties are designed in a way such that for each customer the best rational strategy is to report his true cost distribution type (*Incentive Compatibility*) and the expected net revenue of participating in such DR program is non-negative (*Individual Rationality*). Moreover, we provide both theoretical analysis and numerical studies to demonstrate that our designed mechanism achieves high DR reliability. That is, the deviation between the total reduced demands and the targeted amount is very small. Lastly, we also compare our mechanism with another two mechanisms to show the effectiveness of our MD in terms of the high reliability.

The paper is organized as follows. In Section II, we introduce our DR setup and some basic MD theory. Section III

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presents the outcome-contingent mechanism and discusses the intuition behind it. In Section IV, we give an analytical upper bound for the deviation between the reduced demands and the targeted amount. In Section V, we introduce another two mechanisms for comparison purpose. Lastly, Section VI provides numerical studies.

## II. Problem formulation & Preliminaries

### A. Problem setup

We consider a situation where there is a forecast supply deficit of electricity in the future and a DR aggregator would like to call upon customers to shed loads in order to clear the deficit. In particular, we consider that the total deficit is $M$ units and each customer is able to shed one unit of load. The set of customers is denoted by $\mathcal{N} = \{1, \ldots, N\}$. Our demand response (DR) setup is motivated by day-ahead DR experiments conducted by PGE and PJM [7] [9]. The demand response is operated at two stages:

- at stage I the DR aggregator chooses a subset of customers $J \subset \mathcal{N}$ and asks them to shed a unit of load at stage II;
- at stage II, each customer $i \in J$ decides whether to shed the unit of load or not, depending on his cost of load shedding, denoted by $C_i$.

A main challenge of the DR is the uncertainty of customer costs in load shedding. Before stage II, it is difficult, if not impossible, to know an exact value of the cost even for the customer himself. If the set of customers $J$ is badly chosen and no one sheds the load at stage II, then the system will suffer a serious reliability issue, i.e., none of the deficit will be cleared. This motivates the work of this paper–to design a mechanism where the DR aggregator is able to select the right set of customers to minimize the deviation between the total reduced loads and the targeted level $M$.

Specifically, we model customer $i$'s cost of load shedding as a nonnegative random variable $C_i$. We further parametrize the probability distribution of $C_i$ by assuming that $\{C_i\}_{i\in N}$ follows a similar pattern, i.e., $C_i \sim F(\boldsymbol{\lambda}_i)$ where $F(\cdot)$ denotes a common distribution class and $\boldsymbol{\lambda}_i$ is an individual parameter for customer $i$. $\boldsymbol{\lambda}_i$ is assumed to be privately known to customer $i$. We call $\boldsymbol{\lambda}_i$ as the private type of customer $i$. Without causing any confusion, we will abuse notations by using $C_i(\boldsymbol{\lambda}_i)$ to denote the random variable $C_i$.

**Example 1.** *Suppose $F$ stands for the uniform distribution. Agent $i$'s cost distribution can be formulated as $\mu_i + U[-\sigma_i, \sigma_i]$. The private type of agent $i$ is $\boldsymbol{\lambda}_i = (\mu_i, \sigma_i)$ which is the mean value and variance.*

The mechanism we consider in this paper adopts an outcome-contingent payment plan. At stage I, the aggregator requests each customer to report its private type $\boldsymbol{\lambda}_i$. Based on the reported information, the aggregator selects the group $J$ of customers. A base reward $w$ is given to everyone in the group $J$ to incentivize customers to participate in the DR program. At stage II, customers will decide whether or not to shed a unit of load based on the realization of the cost $C_i$. We use $a_i$ to denote the action of customer $i$: $a_i = 1$ means that customer $i$ sheds a unit of load and $a_i = 0$ means otherwise. If customer $i$ chooses not to shed the load, a penalty $m_i$ will be exerted on the customer. Given this framework, the utility of customer $i$ in the selected group $J$ is given by,

$$u_i(a_i) = w - C_i a_i - m_i(1 - a_i). \tag{1}$$

**Remark 1.** *Assuming that $\{C_i\}_{i\in N}$ follow a similar pattern limits the generality of our results, but our mechanism does not require any further assumption on the distribution $F(\cdot)$. Moreover, compared with using non-parametrized distributions $C_i$ as the bidding type, the DR aggregator can use the parameterized distribution $C_i \sim F(\boldsymbol{\lambda}_i)$ to reduce the complexity of a DR mechanism or auction. The type of probability distribution $F$ can be obtained or approximated using real data.*

Each customer is assumed to be rational when they make decisions. Thus, at stage II, customer $i$ will choose $a_i^* := \arg\max_{a_i\in\{0,1\}} u_i(a_i)$. Because of the randomness of $C_i$, $a_i^*$ and $u_i(a_i^*)$ are random variables as well. Denote the expected utility as $\tilde{u}_i := \mathbb{E}_{C_i\sim F(\boldsymbol{\lambda}_i)} u_i(a_i^*)$. As it is a function of $\boldsymbol{\lambda}_i$, $w$, and $m_i$, we denote the expected utility as $\tilde{u}_i(\boldsymbol{\lambda}_i, w, m_i)$, which can be calculated using the formulation in the following lemma. In addition, for the ease of notation, we will use $a_i$ to denote the $a_i^*$ in the rest of the paper.

**Lemma 1.** *Given customer $i$'s type $\boldsymbol{\lambda}_i$, the based reward $w$ and a penalty $m_i$, the expected utility $\tilde{u}_i$ is given by*[1]

$$\tilde{u}_i(\boldsymbol{\lambda}_i, w, m_i) = w - \int_0^{m_i} Pr(C_i > t|\boldsymbol{\lambda}_i)dt \tag{2}$$

*Proof.*

$$\begin{aligned}
&\tilde{u}_i(\boldsymbol{\lambda}_i, w_i, m_i) = w - m_i Pr(C_i > m_i|\boldsymbol{\lambda}_i) \\
&- \int_0^{m_i} C_i Pr(C_i = t|\boldsymbol{\lambda}_i)dt \\
&= w - m_i + \int_0^{m_i} Pr(C_i \le t|\boldsymbol{\lambda}_i)dt \\
&= w - \int_0^{m_i} Pr(C_i > t|\boldsymbol{\lambda}_i)dt
\end{aligned}$$

□

**Remark 2.** *The assumption $a_i \in \{0, 1\}$ can be satisfied by adopting proper payment rules. For example, if customer $i$ is paid the same for $a_i \in [0, 1)$, then the rational action for customer $i$ with $C_i \ge 0$ is either $a_i = 0$ or $a_i = 1$.*

In our proposed mechanism, the base reward $w$ is fixed and is the same for all customers in the selected group $J$. The fixed rate ensures a bounded total budget that is used to pay for the DR, and the same rate is used to ensure fairness. However, in order to incentivize customers to report truthful information $\boldsymbol{\lambda}_i$, individual $m_i$ will be determined for each customer $i \in J$. The mechanism will be introduced in details

[1] Without loss of generality, we only consider continuous cost distributions in the rest of the paper. All results and analysis hold for discrete cost distributions with a minor change, i.e., replacing integrals with summations.

in the next section. Here, we provide some discussion on the constraints when determining $m_i$. From the expression of $\tilde{u}_i$ in (2), the expected utility $\tilde{u}_i$ is a monotonically decreasing function on the penalty $m_i$. In order to incentivize customers to participate in the DR program, customers' expected utility should be guaranteed to be nonnegative. This is know as *individual rationality (IR)* in mechanism design (MD) literature. As a result, $m_i$ should be lower than the maximum penalty $\tilde{m}_i$ for each $i \in J$,

$$\tilde{m}_i := \arg\max_{m_i} \{m_i | \tilde{u}(\boldsymbol{\lambda}_i, w, m_i) \geq 0\} \tag{3}$$

It is easy to check that for any $m_i \leq \tilde{m}_i$, the expected utility of customer $i$ is nonnegative.

As discussed before, the main objective of this paper is to design a mechanism to ensure a small deviation between the total reduced loads and the targeted amount $M$ units. The smaller the deviation is, the higher reliability the system has. Mathematically, we define the reliability index $R$ as the following,

**Definition 1.** *The reliability index $R$ is defined as the inverse of the deviation, i.e.,*

$$R^{-1} = \sqrt{\mathop{\mathbb{E}}_{\vec{a}, \vec{\boldsymbol{\lambda}}} (M - \sum_{i \in J} a_i)^2} \tag{4}$$

*where $\vec{a} = \{a_1, \ldots, a_N\}$ and $\vec{\boldsymbol{\lambda}} = \{\boldsymbol{\lambda}_i, \ldots, \boldsymbol{\lambda}_N\}$.*

In this definition, the expectation is taken over the uncertain $a_i$ and $\boldsymbol{\lambda}_i$. Here we assume that even though the exact customer type $\boldsymbol{\lambda}_i$ is private information, $\boldsymbol{\lambda}_i$ is drawn randomly from a publicly known distribution. This assumption is borrowed from Bayesian analysis [16]. The particular form of the reliability in (4) is motivated by the standard deviation definition in statistics.

### B. Preliminaries: A brief introduction on mechanism design (MD)

Before we proceed with our designed DR mechanism, here we provide a brief introduction on general MD.

In a MD problem, there is a set of outcomes $O$, and a set of self-interested agents $\{1, \ldots, N\}$ each with a private cost function $c_i(o)$ where $o \in O$. A typical mechanism consists of three parts:[2]

- collecting reports or bids $b_i$ on the costs $c_i(\cdot)$ from agents,
- an outcome rule $o(\vec{b})$ selecting outcomes based on the bid profile $\vec{b} = \{b_1, \ldots, b_N\}$,
- a payment rule $p_i(\vec{b}, o(\vec{b}))$ based on the agent reports and the selected outcome.

In the mechanism, agent $i$'s utility function is expressed as $u_i(\vec{b}, c_i) = p_i(\vec{b}, o(\vec{b})) - c_i(o(\vec{b}))$. Agents are assumed to be rational and bid to maximize their utilities. A bidding strategy $s_i$ specifies a mapping from private cost $c_i$ to a bid $b_i$.

[2]Strictly speaking, this is called as a mechanism with money.

**Definition 2.** *A bidding strategy profile $(s_1^*, \ldots, s_N^*)$ is called a dominant strategy equilibrium (DSE) if $s_i^*$ is an optimal strategy for agent $i$ no matter what strategies other agents are playing,*

$$u_i(s_i^*(c_i), s_{-i}(c_{-i}), c_i) \geq u_i(s_i(c_i), s_{-i}(c_{-i}), c_i)$$

*for any $i$, $s_i$, $s_{-i}$, $c_i$ and $c_{-i}$. Here the index $-i$ means all agents excluding $i$.*

Depending on the mechanism, the reports/bids $b_i$ can be directly the privation information $c_i$ or be other indirect information which is based on the private information. The former case is called as a *direct-revelation mechanism (DRM)*. One major goal of MD is to elicit truthful private information from customers. This is called as *incentive compatibility* (IC). The formal definition is provided as follows,

**Definition 3.** *A DRM is incentive-compatible (IC) if truthful reporting strategy $s_i(c_i) = c_i$ is a DSE.*

*Revelation Principle Theorem* [17] tells that focusing on IC DRM does not lose any generality.

**Theorem 1.** *Any mechanism with a DSE can be implemented as an IC DRM.*

To provide incentives for participation, a mechanism needs to have the property of *individual rationality (IR)*,

**Definition 4.** *A DSE in a MD is individual rational (IR) if it guarantees nonnegative utility for all agents.*

An IC DRM is called IR if truthful reporting is IR. We will focus on designing an IC and IR DRM in this paper.

## III. Mechanism

In this section, we first discuss the intuition behind our mechanism regarding the reliability and IR property. We then present our mechanism and discuss more-formally why our mechanism is IC and IR and how our mechanism achieves high reliability.

For the reliability analysis, we split the squared deviation, which is $(R^{-1})^2$, into two terms Error1 and Error2 as shown below and analyze them respectively,

$$\begin{aligned}
(R^{-1})^2 &= \mathop{\mathbb{E}}_{\vec{\boldsymbol{\lambda}}} \left[ \mathop{\mathbb{E}}_{\vec{a}} (M - \sum_{i \in J} a_i)^2 | \vec{\boldsymbol{\lambda}} \right] \\
&= \mathop{\mathbb{E}}_{\vec{\boldsymbol{\lambda}}} (M - \sum_{i \in J} Pr(a_i = 1 | \vec{\boldsymbol{\lambda}}))^2 \\
&+ \mathop{\mathbb{E}}_{\vec{\boldsymbol{\lambda}}} \sum_{i \in J} Pr(a_i = 1 | \vec{\boldsymbol{\lambda}})(1 - Pr(a_i = 1 | \vec{\boldsymbol{\lambda}})) \\
&= \underbrace{\mathop{\mathbb{E}}_{\vec{\boldsymbol{\lambda}}} (M - \sum_{i \in J} Pr(C_i \leq m_i | \vec{\boldsymbol{\lambda}}))^2}_{\text{Error1}} \\
&+ \underbrace{\mathop{\mathbb{E}}_{\vec{\boldsymbol{\lambda}}} \sum_{i \in J} Pr(C_i \leq m_i | \vec{\boldsymbol{\lambda}})(1 - Pr(C_i \leq m_i | \vec{\boldsymbol{\lambda}}))}_{\text{Error2}}
\end{aligned}$$

To achieve high reliability, both Error1 and Error2 should be kept small.

Error1 mainly depends on $\sum_{i\in J} Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})$, which is the total reduced loads by all customers in expectation. For small Error1, the expected total reduction should be close to $M$ units. This is difficult because the probability $Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})$ is unavailable to the DR aggregator as the cost distribution types are private to customers. Therefore, a mechanism should elicit the true private information and provide a good estimation to $Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})$ accordingly. In addition, a mechanism should ensure the estimated $\sum_{i\in J} Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})$ is close to $M$.

Error2 relies on the number of customers in $J$ and the probability of load shedding $Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})$. A mechanism can keep a small Error2 by i) selecting fewer customers and ii) incentivizing customers to respond with a higher probability. Note that i) conflicts with the requirement in Error1 since a few customers may not be sufficient to make $\sum_{i\in J} Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})$ close to $M$. Hence a tradeoff should be made between Error1 and Error2. As for ii), one way to incentivize customers to respond is to have high penalties. However, if penalties are set too high, the MD will not guarantee IR. As discussed in Section II, IR requires $m_i$ to be no greater than the maximum penalty $\tilde{m}_i$ for each $i$. This shows that there is a trade-off between reliability and IR as well.

### A. Mechanism

We are ready to present our mechanism below. At stage I, the DR aggregator will perform the following procedure. (Ties are broken randomly.)

- (Bids) The DR aggregator asks everyone to report their types. The reports may not be truthful. Denote the reported type as $\boldsymbol{\lambda}'_i$ for customer $i \in \mathcal{N}$. The report profile is denoted by $\vec{\boldsymbol{\lambda}}'$.
- (Allocation/Selection rule) The aggregator relabels customers by $\tilde{m}_i$ in a decreasing order: $\tilde{m}_1 \geq \cdots \geq \tilde{m}_n$, then the aggregator selects the group $J$ using the following index $k$,

$$\begin{aligned}&k = \arg\min_j \{j | \textstyle\sum_{i=1}^{j} Pr(C_i \leq m'_i|\vec{\boldsymbol{\lambda}}') \geq M - \beta\}\\ &m'_i = \tilde{m}_k\end{aligned} \tag{5}$$

  where $\beta$ is chosen to be $1/2$ with reasons provided later in Section III-B. The group $J$ is selected according to:

$$\begin{aligned}i \in J, &\quad \text{if } i \leq k\\ i \notin J, &\quad \text{otherwise}\end{aligned}$$

- (Payment rule) If agent $i \in J$, its payment involves two parts: $w$ and $m_i$, where $w$ is a fixed reward and $m_i$ is determined by the following rule. Excluding $i$, do the same selection rule again and find the index $k'(i)$ of the last agent selected. $m_i$ is given by,

$$m_i = \tilde{m}_{k'(i)}$$

When there are not enough customers, there might be cases when the mechanism terminates at $N$ before finding a proper $k$ in the selection rule or $k'(i)$ in the payment rule for some customer $i$. For those cases, we let $k = N$ or $k'(i) = N$. When $k = N$, penalties are set to be 0. In Section VI, we show that it only takes a few customers to be sufficient to avoid these cases. Therefore, in the following analysis, we assume that customers are enough for implementing the selection and payment rules.

### B. Discussion: Properties of our mechanism

Firstly, the following theorem proves that our mechanism is both IC and IR. Hence, customers report their true distribution types and are willing to join the DR voluntarily.

**Theorem 2.** *The mechanism described above is IC and IR.*

*Proof.* If $x_i(\vec{\boldsymbol{\lambda}}') = 1$, the payment $w$ and $m_i$ are independent of agent i's report $\boldsymbol{\lambda}'_i$. If $x_i(\vec{\boldsymbol{\lambda}}') = 0$, there will be no payment. Therefore, agent's utility remains the same with different reports with the same outcome.

Therefore, we only need to check that the allocation outcome from truthful reporting maximizes agents' utilities. If $x_i(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}'_{-i}) = 1$, $m_i = \tilde{m}_{k'(i)}$. Notice that $k'(i) > i$ so $\tilde{m}_{k'(i)} \leq \tilde{m}_i$, which means $m_i \leq \tilde{m}_i$, thus $\tilde{u} = w - \int_0^{m_i} 1 - Pr(C_i \leq t)dt \geq w - \int_0^{\tilde{m}_i} 1 - Pr(C_i \leq t)dt = 0$. Therefore, agent i's utility by truthful reporting is no worse than misreporting to be unselected. If $x_i(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}'_{-i}) = 0$, the allocation will stop at $k$ before reaching agent i. In this case, the punishment should be at least $\tilde{m}_k$ which is no less than $\tilde{m}_i$. With the same argument, the utility by deviating is nonpositive. If agent $i$ is within a tie, similar analysis will show that $\tilde{u} = 0$ no matter how agent $i$ reports. In total, there is no useful misreport for agents. Therefore, the mechanism is IC. IR has already been shown above. □

Because the mechanism is IC, in the rest of the paper, we will let $\boldsymbol{\lambda}' = \boldsymbol{\lambda}$. Next we discuss why the mechanism is able to achieve high reliability by following the discussion at the beginning of this section.

Error1 is ensured to be small mainly because of the selection rule. Firstly, we note that if the penalty $m_i$ is chosen to be $\tilde{m}_k$ as in (5), then the expected total reduced loads $\sum_{i=1}^{k} Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})$ is guaranteed to be within the interval $[M - \beta, M + (1 - \beta)]$ where $\beta$ is chosen to be $1/2$ to minimize $\max(1 - \beta, \beta)$. Though in order to ensure the IC property, the penalty $m_i$ is chosen to $\tilde{m}_{k'(i)}$ in the payment rule, $\tilde{m}_k$ and $\tilde{m}_{k'(i)}$ are not very different from each other when the number of customers $N$ is relatively large. This is formally proven in Section IV and further numerically verified in Section VI. As a result, the expected total reduced demands $\sum_{i=1}^{k} Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})$ is close to $M$, meaning that Error1 is small.

For Error2, the selection rule achieves a good tradeoff between Error1 and Error2 by selecting the least number of customers to ensure small Error1, and the payment rule achieves a good tradeoff between a small Error2 and IR by selecting a high enough penalty without exceeding $\tilde{m}_i$.

**Remark 3.** *Our mechanism uses the maximum penalties as a criterion to select customers. Though the maximum penalty reflect the customer's probability of load shedding, it is also* *affected by other factors such as the mean value of the cost. As a result, our mechanism may miss some good customers with high probability of load shedding. However, it is difficult to design an IC and IR mechanism if aiming to select these customers. Future work includes investigating other possible mechanisms.*

## IV. Theoretical Guarantee on Reliability

In this section, we will provide an analytical upper bound for the deviation. To obtain the bound, we make the following assumptions and introduce a few useful definitions and lemmas.

### A. Main assumptions

The first assumption is on the distribution of private types.

**Assumption 1.** *Agents' types $\{\boldsymbol{\lambda}_i\}_{i \in N}$ are independent and identically distributed (i.i.d.) with a continuous distribution $H$. The support set of $H$, denoted by $T$, is compact.*

Assumption 1 ensures that customer maximum penalties $\tilde{m}_i$ are i.i.d.. As a result, $\tilde{m}_1, \ldots, \tilde{m}_N$ with new labels are order statistics where $\tilde{m}_k$ denotes the $k$th largest maximum penalty. In the rest of this section, we will omit subscript $i$ in the notations as customers are i.i.d, except when dealing with order statistics. For example, we will use $\tilde{m}(\boldsymbol{\lambda})$ to denote the maximum penalty for a customer with distribution type $\boldsymbol{\lambda}$.

Next we assume function continuities.

**Assumption 2.** *$Pr(C(\boldsymbol{\lambda}) \le m|\boldsymbol{\lambda})$ is continuous with respect to $(m, \boldsymbol{\lambda})$ . $\tilde{m}(\boldsymbol{\lambda})$ is continuous with $\boldsymbol{\lambda}$.*

Assumption 1 and 2 imply that $\tilde{m}$ is bounded.

**Lemma 2.** *$\tilde{m}$ has maximum $\bar{m}$ and minimum $\underline{m}$.*

In addition, we assume that customers have a nonzero probability of load shedding at least in some scenarios.

**Assumption 3.**

$$\begin{aligned} \mathbb{P} \quad &:= \quad \max\nolimits_{\boldsymbol{\lambda} \in T, m \le \tilde{m}(\boldsymbol{\lambda})} Pr(C(\boldsymbol{\lambda}) \le m|\boldsymbol{\lambda}) \\ &= \quad \max\nolimits_{\boldsymbol{\lambda} \in T} Pr(C(\boldsymbol{\lambda}) \le \tilde{m}(\boldsymbol{\lambda})|\boldsymbol{\lambda}) > 0 \end{aligned}$$

This assumption is equivalent to requiring the reward $w$ to be large enough so that at least some customers are incentivized to perform DR, as claimed in Lemma 3.

**Lemma 3.** *$\bar{m} > w$ is equivalent to $\mathbb{P} > 0$.*

Although it seems that $\bar{m} > w$ requires $w$ to be small enough, it is actually the opposite way because $\bar{m}$ implicitly depends on $w$ and this inequality only holds when $w$ is large enough. The following example helps explain this counter-intuitive statement.

**Example 2.** *Consider $C \sim U[4, 9]$. Figure 1 shows the relation between $w$ and $\tilde{m}$ together with $Pr(C \le \tilde{m}|\boldsymbol{\lambda})$.*[3] *When $w = 3$, $\tilde{m} = 3$ and $Pr(C \le \tilde{m}|\boldsymbol{\lambda}) = 0$. If $w$ is increased to 6, $\tilde{m} \approx 6.76$, and $Pr(C \le \tilde{m}|\boldsymbol{\lambda}) \approx 0.55 > 0$. The critical value of $w$ is 4. When $w > 4$, we have $\tilde{m} > w$ and $Pr(C \le \tilde{m}|\boldsymbol{\lambda}) > 0$.*

[3]Here we only have one type of $\boldsymbol{\lambda}$, i.e., the cardinality $|T| = 1$.

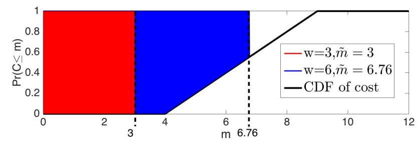



Fig. 1: $Pr(C < \tilde{m}(\boldsymbol{\lambda})|\boldsymbol{\lambda})$ with different $w$

Note that Assumptions 1-3 are satisfied by many distribution classes, e.g., exponential distribution and uniform distribution. We use these two distribution classes as examples to illustrate the assumptions.

**Example 3.** *Consider $F$ as exponential distribution. Each customer $i$ has a cost $C_i = \mu_i + \sigma_i X_i$ where $\boldsymbol{\lambda}_i = (\mu_i, \sigma_i)$, $\mu_i \sim U[15, 20]$, $\sigma \sim U[5, 10]$ and $X_i$ i.i.d. from $Exp(1) - 1$. Here $\mu_i$ is the mean cost and $\sigma_i$ is the cost variance. The base reward $w = 14$.*

*The following two functions are continuous thus satisfying Assumption 2.*

$$\tilde{m}_i(\boldsymbol{\lambda}_i) = \begin{cases} 14, & \mu_i - \sigma_i \ge 14 \\ \mu_i - \sigma_i + \sigma_i \log(\frac{\sigma_i}{\mu_i - 14}), & \mu_i - \sigma_i < 14 \end{cases}$$

$$Pr(C \le m|\boldsymbol{\lambda}) = \begin{cases} 1 - \exp(-1 + \frac{\mu - m}{\sigma}), & m \ge \mu - \sigma \\ 0, & m \le \mu - \sigma \end{cases}$$

*$\bar{m} = 5 + 10\log(10) > w$. According to Lemma 3, Assumption 3 is satisfied.*

**Example 4.** *Consider $F$ to be uniform distribution. Each customer $i$ has a cost $C_i = \mu_i + \sigma_i X_i$ where $\boldsymbol{\lambda}_i = (\mu_i, \sigma_i)$, $\mu_i \sim U[16, 20]$, $\sigma \sim U[12, 16]$ and $X_i$ i.i.d. from $U[-1, 1]$. Here $\mu_i$ is the mean cost and $\sigma_i$ is the cost variance up to a scale factor. The base reward $w = 15$.*

*The following two functions are continuous thus satisfying Assumption 2.*

$$\tilde{m}(\boldsymbol{\lambda}) = \begin{cases} 15, & \mu - \sigma \ge 15 \\ \mu + \sigma - 2\sqrt{\sigma}\sqrt{\mu - 15}, & \mu - \sigma < 15 \end{cases}$$

$$Pr(C \le m|\boldsymbol{\lambda}) = \begin{cases} 0, & m < \mu - \sigma \\ \frac{m - \mu}{2\sigma} + 1/2, & \mu - \sigma \le m \le \mu + \sigma \\ 1, & m > \mu + \sigma \end{cases}$$

*Assumption 3 is satisfied because $\bar{m} = 24 > w$.*

### B. The Deviation Upper Bound

Next, we introduce some notations and properties that will be used in the main result of the deviation upper bound in Theorem 3.

**Definition 5.** *For any $\epsilon > 0$, we define a compact set*

$$\Gamma(\epsilon) = \{\boldsymbol{\lambda} \in T | \tilde{m}(\boldsymbol{\lambda}) \ge \bar{m} - \epsilon\}$$

Since our mechanism selects agents with large $\tilde{m}$, the types of selected customers are mostly in set $\Gamma(\epsilon)$ when the number of customers $N$ is large. The next lemma gives a characterization on $Pr(C(\boldsymbol{\lambda}) \leq m|\boldsymbol{\lambda})$ when $\boldsymbol{\lambda} \in \Gamma(\epsilon)$ and $\bar{m} - \epsilon \leq m \leq \bar{m}$.

**Lemma 4.** *We have* $0 < \underline{p}^{\bar{m}-\epsilon} \leq Pr(C(\boldsymbol{\lambda}) \leq m|\boldsymbol{\lambda}) \leq \bar{p}^{\bar{m}-\epsilon}$ *for any* $0 < \epsilon < \bar{m} - w$*, any* $\boldsymbol{\lambda}$ *with* $\tilde{m}(\boldsymbol{\lambda}) \geq \bar{m} - \epsilon$ *and* $m \in [\bar{m}-\epsilon, \bar{m}]$*, where* $\underline{p}^{\bar{m}-\epsilon} := \min_{\boldsymbol{\lambda}\in\Gamma(\epsilon)} Pr(C \leq \bar{m}-\epsilon|\boldsymbol{\lambda})$ *and* $\bar{p}^{\bar{m}-\epsilon} := \max_{\boldsymbol{\lambda}\in\Gamma(\epsilon)} Pr(C \leq \bar{m}|\boldsymbol{\lambda})$

Given an $\epsilon > 0$, we define $d(\epsilon)$ as

$$d(\epsilon) := \max_{\boldsymbol{\lambda}\in\Gamma(\epsilon)} Pr(C \leq \bar{m}|\boldsymbol{\lambda}) - Pr(C \leq \bar{m} - \epsilon|\boldsymbol{\lambda}). \quad (6)$$

In the following, we present the upper bound of the deviation $R^{-1}$.

**Theorem 3.** *For any* $0 < \epsilon < \bar{m} - w$*, we have*

$$R^{-1} \leq \sqrt{\eta_\epsilon + [N/4 + M^2 - \eta_\epsilon]Pr(A)} \quad (7)$$

*where*

$$\eta_\epsilon = K_\epsilon \xi(\bar{p}^{\bar{m}-\epsilon}, \underline{p}^{\bar{m}-\epsilon}) + (K_\epsilon d(\epsilon) + 1/2)^2$$
$$K_\epsilon = \lceil \frac{M - 1/2}{\underline{p}^{\bar{m}-\epsilon}} \rceil$$
$$A = \{\tilde{m}_{K_\epsilon+1} < \bar{m} - \epsilon\}$$
$$Pr(A) = G(\bar{m} - \epsilon)^N \sum_{i=0}^{K_\epsilon} \binom{N}{i} \left(\frac{1 - G(\bar{m} - \epsilon)}{G(\bar{m} - \epsilon)}\right)^i$$
$$G(\bar{m} - \epsilon) = Pr(\tilde{m} < \bar{m} - \epsilon)$$

*and*

$$\xi(\bar{p}^{\bar{m}-\epsilon}, \underline{p}^{\bar{m}-\epsilon}) = \begin{cases} \underline{p}^{\bar{m}-\epsilon}(1 - \underline{p}^{\bar{m}-\epsilon}) & \text{If } \frac{1}{2} < \underline{p}^{\bar{m}-\epsilon} \\ \bar{p}^{\bar{m}-\epsilon}(1 - \bar{p}^{\bar{m}-\epsilon}) & \text{If } \bar{p}^{\bar{m}-\epsilon} < \frac{1}{2} \\ 1/4, & \text{otherwise} \end{cases}$$

A key lemma to the proof of this theorem is given below and the complete proof is saved for Appendix. The lemma states it is of high probability that the number of customers selected is bounded and the difference is small between real penalty $\tilde{m}_{k'(i)}$ and estimated penalty $\tilde{m}_k$.

**Lemma 5.** $\forall 0 < \epsilon < \bar{m} - w$*, event* $A^c$ *being* $\{\tilde{m}_1 \geq \cdots \geq \tilde{m}_{K_\epsilon+1} \geq \bar{m} - \epsilon\}$*, and event* $B = \{k \leq K_\epsilon, k'(i) \leq K_\epsilon + 1, \forall i \leq k\}$*, we have* $A^c \subseteq B$*.* $Pr(A) = \sum_{i=0}^{K_\epsilon} \binom{N}{i}(1 - G(\bar{m} - \epsilon))^i G(\bar{m} - \epsilon)^{N-i}$*.* $K_\epsilon = \lceil \frac{M-1/2}{\underline{p}^{\bar{m}-\epsilon}} \rceil$

### C. Effects of $N$, $\epsilon$ and $M$ on the Upper Bound in (7)

Now we use the upper bound in (7) to discuss the effects of $N$, $\epsilon$, and $M$ on the deviation. Because $\eta_\epsilon$ is independent of $N$, the number of customers only affects the part $[N/4 + M^2 - \eta_\epsilon]Pr(A)$. The following lemma shows that this part vanishes when $N$ becomes very large. Therefore, $R^{-1}$ can roughly be bounded by $\sqrt{\eta_\epsilon}$ when $N$ is large.

**Lemma 6.** *When* $\epsilon > 0$*,*

$$\lim_{N\to\infty} [N/4 + M^2 - \eta_\epsilon]Pr(A) = 0 \quad (8)$$

The upper bound in (7) holds for all $\epsilon \in (0, \bar{m} - w)$, which gives much flexibility in choosing $\epsilon$. When choosing a very small $\epsilon$, $\Gamma(\epsilon)$ shrinks, resulting in a large $\underline{p}^{\bar{m}-\epsilon}$ and a small $\bar{p}^{\bar{m}-\epsilon}$. Moreover, $d(\epsilon)$ becomes very small as shown in the following Lemma.

**Lemma 7.** *We have* $\lim_{\epsilon\to 0} d(\epsilon) = 0$*.*

As a result, $\xi(\bar{p}^{\bar{m}-\epsilon}, \underline{p}^{\bar{m}-\epsilon})$ and $K_\epsilon$ become very small too, which leads to a very small $\eta_\epsilon$. Therefore, when $N$ is very large, a small $\epsilon$ provides a tighter upper bound. However, tiny $\epsilon$ make $G(\bar{m} - \epsilon)$ closer to 1 resulting in a slower vanishing rate of (8). Hence, when $N$ is not very large, choosing bigger $\epsilon$ may provide better bounds by making $[N/4 + M^2 - \eta_\epsilon]Pr(A)$ smaller.

Finally, we present a bound for infinite $N$ and discuss about $M$'s effect.

**Corollary 1.** *When the number of agents is infinite,*

$$\lim_{N\to+\infty} R^{-1} \leq \sqrt{(\frac{M - 1/2}{\underline{p}^{\bar{m}}} + 1)\xi(\bar{p}^{\bar{m}}, \underline{p}^{\bar{m}}) + 1/4} \quad (9)$$

The upper bound in (9) is $O(\sqrt{M})$, thus the relative deviation $R^{-1}/M$ is $O(\frac{1}{\sqrt{M}})$. This indicates that larger numerical value of $M$ will produce a lower relative deviation when $N$ is large.

In section VI, we will use numerical studies to further confirm the effects of $N$ and $M$ on the reliability.

## V. Other Mechanisms for Comparison

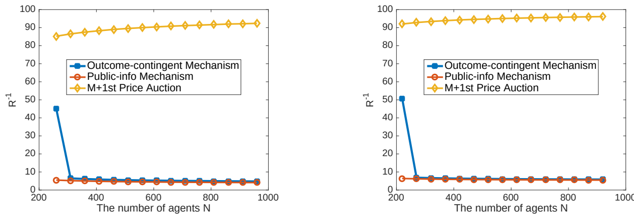
For the purpose of comparison, this section introduces two other mechanisms: public-info mechanism and $(M + 1)$st price auction. Public-info mechanism assumes that the customer cost distribution types are public knowledge. Though this assumption is unrealistic, the mechanism provides a near-optimal solution for reliability and thus will be a good benchmark when accessing reliability. Comparison with this mechanism demonstrates the reliability loss of our mechanism due to the private costs. $(M + 1)$st price auction is a commonly used mechanism in the case with determinant costs. Applying this to a system with random costs will show the consequence of neglecting cost uncertainties. The numerical comparisons are provided in the next section VI.

### A. Public-info mechanism

In the public-info mechanism, it is assumed that the DR aggregator knows the cost distributions of all customers. For each customer $i$, the DR aggregator calculates the maximum penalty $\tilde{m}_i$ and probability of load shedding with the maximum penalty $Pr(C_i \leq \tilde{m}_i|\boldsymbol{\lambda}_i)$. Customers are re-indexed in the decreasing order of $Pr(C_i \leq \tilde{m}_i|\boldsymbol{\lambda}_i)$.

- (Allocation/Selection rule): The DR aggregator selects first $k$ customers such that the total expected load reduction reaches $M - 1/2$ for the first time. Let $k = N$ if $k$ is not found before termination.
- (Payment rule): Customer $i$ that is selected receives a base reward $w$ and needs to pay a penalty $\tilde{m}_i$ if he does not respond at stage II when DR is needed.

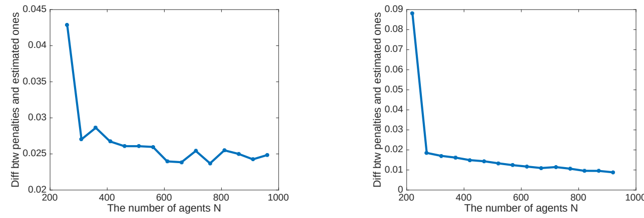
Intuitively speaking, the maximum penalty forces all customers to perform load shedding to the largest extent without violating IR, and the allocation rule priorly selects customers




(a) Exponential Costs

(b) Uniform Costs

Fig. 2: The relation between $N$ and $R^{-1}$ of our mechanism with private information, and $R^{-1}$ of public-info mechanism




(a) Exponential Costs

(b) Uniform Costs

Fig. 3: The relation between the number of agents $N$ and the average $|\tilde{m}_k - \tilde{m}_{k'(i)}|$

with the highest probabilities of response in order to ensure high reliability. Thus the public-info mechanism provides a near-optimal DR plan, which is unrealistic to implement in practice due to private costs.

### B. $(M+1)$st price auction

Most of the current DR plans assume costs are determinant. This setting is analogous to the classic multi-item auction scenario where $N$ customers bid for $M$ units. It is assumed that the cost $C_i$ is a fixed and determinant value. We apply the $(M+1)$st price auction, which is a generalization of second-priced auction, to this setting. In the $(M+1)$st price auction, the DR aggregator collects $C_i$ from all customers. Customers are re-indexed with an increasing order of reports $C_i$. (Ties are broken randomly and $N > M$ is assumed for simplicity.)

- (Allocation/Selection rule): The DR aggregator selects first $M$ customers.
- (Payment rule): Customers that are selected receives a payment that is equal to $C_{M+1}$, i.e. the report of the $(M+1)$st customer if he reduces one unit of load and zero otherwise.

This auction is IC and IR when $C_i$ is determinant for all $i$ [17]. Moreover, all $M$ customers will shed loads at stage II due to utility maximization. Thus the auction provides good reliability in a world without random costs.

When applying this auction to reality, cost uncertainties will results in the following bidding strategy equilibrium:

**Theorem 4.** *Suppose $\underline{C}_i$ is the largest lower bound such that $Pr(C_i \geq \underline{C}_i) = 1$ for each $i$. It is a DSE when customer $i$ bids $\underline{C}_i$ for each $i$.*

The theorem shows that customers tend to report their minimal possible cost in order to be selected. This leads to a low payment as well. As a result, there are not enough incentives for customers to shed loads, resulting in poor reliability as shown in Section VI.

## VI. Numerical Results

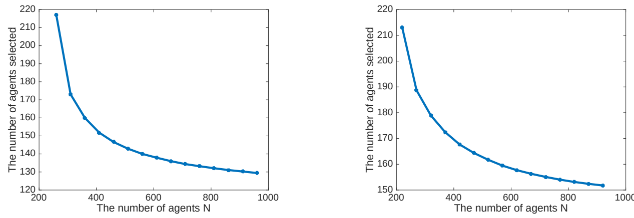
In this section, we provide numerical results to demonstrate the high reliability achieved by our mechanism. We also compare the mechanism with the public-info mechanism and the $(M+1)$st price auction introduced in the previous section. In addition, we study the total budget that is used in the proposed DR mechanism and discuss how to improve the reliability.

We assume each customer $i$ has a nonnegative cost $C_i = \mu_i + \sigma_i X$. Two cost distribution classes for $X$ are considered here, exponential distribution and uniform distribution. In the exponential case, $X \sim Exp(1) - 1$. $\mu_i$ is uniformly random from $[15, 20]$ and $\sigma_i$ is uniformly random from $[5, 10]$. The base reward $w = 14$. In the uniform case, $X \sim U[-1, 1]$. $\mu_i$ is uniformly random from $[16, 20]$ and $\sigma_i$ is uniformly random from $[12, 16]$. The base reward $w = 15$. For each class, we repeat the MD for 800 times in order to obtain a good approximation on the expectation that is used in the reliability assessment (Definition 1).

### A. Reliability Analysis and Comparison with Public-Info mechanism and $(M+1)$st Price Auction

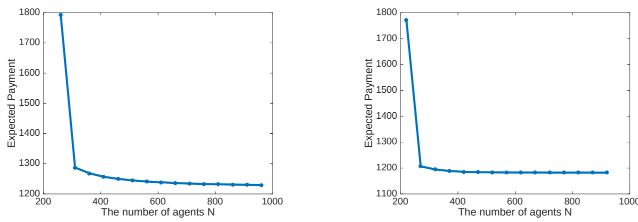
In Figure 2, the targeted DR value $M$ is 100 units, $N$ is from 260 to 960 in exponential-cost case and from 220 to 920 in uniform-cost case. Simulation results show how the deviation between the true value of reduced loads and the targeted value (i.e. the inverse of the reliability, $R^{-1}$) changes when the number of customers $N$ increases. The simulation results are similar for the two distribution cases. Thus, we will only focus on the exponential case to discuss the implication of the simulation results.

Figure 2 shows that when $N = 260$, $R^{-1}$ of our mechanism is large. This is due to the fact that customers are insufficient when $N$ is small. When $N \geq 310$, there are sufficient customers and $R^{-1}$ of our mechanism drops to a low level (less than 7). This means our mechanism provides good reliability as long as customers are enough. In addition, the ratio of sufficient $N$ to $M$ is less than 4 in this case, meaning that a relatively small number of customers are sufficient to ensure high reliability. Furthermore, we see that $R^{-1}$ decreases with $N$, which means that the system reliability can be improved by recruiting more customers in DR. Moreover, we see that when $N$ is sufficient, the reliability of outcome-contingent mechanism and public-info mechanism are almost the same. This indicates that the reliability loss caused by private cost is almost negligible in our designed mechanism. Lastly, Figure 2 tells that the $(M+1)$st price auction performs poorly when applied to the DR with random uncertain costs. This further confirms the importance of considering random costs when designing DR mechanisms.



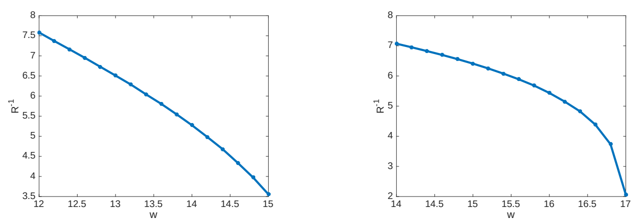

(a) Exponential Costs (b) Uniform Costs

Fig. 4: The relation between the number of agents $N$ and the number of agents selected



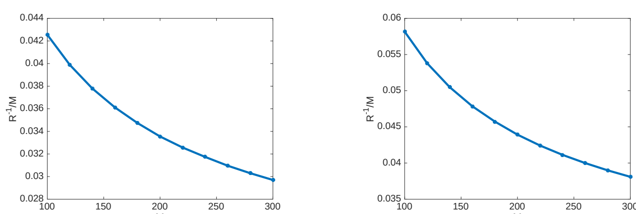

(a) Exponential Costs (b) Uniform Costs

Fig. 5: The relation between the number of agents $N$ and the total payment




(a) Exponential Costs (b) Uniform Costs

Fig. 6: The relation between the reward $w$ and Error




(a) Exponential Costs (b) Uniform Costs

Fig. 7: The relation between $M$ and Error/M with $N = 2000$.

In Figure 3, we see that the average difference is very small between the real penalty $\tilde{m}_{k'(i)}$ exerted to customers and the estimated one $\tilde{m}_k$ used in allocation rule. In addition, the difference is even less when $N$ increases from 260 to 310. This demonstrates the claim in Section III that $\tilde{m}_{k'(i)}$ and $\tilde{m}_k$ are very similar when $N$ is large.

### *B. Budget Concerns*

To incentivize customers to participate in the DR, the DR aggregator needs to pay customers. If the total payment is too high, it may be not worthwhile to clear the supply deficit through DR programs. Thus here we discuss the budget used in our proposed mechanism. As the mechanism fixes the base reward $w$, one important factor for the budget is the number of customers selected by our mechanism. Figure 4 shows the number of customers selected by our mechanism with different $N$. We see that less customers will be selected when $N$ is larger. This is because when $N$ is large, the number of "good-quality" customers that have a higher probability of load shedding will also be large. In this case our mechanism can select fewer "good-quality" customers instead of more "poor-quality" ones to cover the deficit. Consequently, the expected payment (rewards minus penalties) decreases with increasing $N$ as well. This may look surprising and counterintuitive because a system designer might worry about a higher DR budget when recruiting more customers. However, as long as the mechanism is able to select the right customers, the budget indeed decreases with larger $N$.

### *C. How to improve reliability?*

In Figure 2 of section VI-A, we have shown that reliability can be improved by recruiting more customers in DR. In addition, we will show that higher $w$ and larger numerical value of $M$ also raise reliability. In Figure 6, we fix $N = 600$ and $M = 100$. $w$ changes from 12 to 15 for exponential costs and from 14 to 16 for uniform costs. We see that increasing base reward will reduce $R^{-1}$ thus improving reliability. In Figure 7, we let $M$ varies from 100 to 300 but fix $N = 2000$. We see that the relative deviation $R^{-1}/M$, which is the ratio between deviation and M, decreases with increasing $M$. This suggests one way to improve the reliability for a targeted supply deficit is to divide the deficit into smaller units. For instance, if $M = 100$ KW, setting one unit of loads to be $0.5KW$ will have higher reliability than setting one unit to be $1KW$.

## VII. Conclusion

In this paper, we present a two-stage DR mechanism to handle cost uncertainties of self-interested customers. We define reliability as the inverse of deviation between the total reduced loads and the targeted value. Besides being IC and IR, we show that our mechanism achieves high reliability. Moreover, we show that the reliability can be improved by recruiting more customers, increasing the budget, and dividing deficit into smaller units when adopting our mechanism. Future work includes investigating other possible mechanisms and generalizing the assumption on the cost uncertainties.

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## APPENDIX

**Lemma 2.** *$\tilde{m}$ has maximum $\bar{m}$ and minimum $\underline{m}$.*

*Proof.* This is because $\tilde{m}(\boldsymbol{\lambda})$ is continuous on a compact set $T$. □

**Lemma 3.** *$\bar{m} > w$ is equivalent to $\mathbb{P} > 0$.*

*Proof.* ($\Rightarrow$): Consider $\boldsymbol{\lambda}^*$ such that $\tilde{m}(\boldsymbol{\lambda}^*) = \bar{m}$. By definition,

$$w = \int_0^{\bar{m}} Pr(C > t|\boldsymbol{\lambda}^*)dt$$

Since $w < \bar{m}$, we have $Pr(C > \bar{m}|\boldsymbol{\lambda}^*) < 1$, otherwise $Pr(C > t|\boldsymbol{\lambda}^*) = 1$ for $0 \leq t \leq \bar{m}$ leading to $\bar{m} = w$. Therefore,

$$\max_{\boldsymbol{\lambda} \in T} Pr(C(\boldsymbol{\lambda}) \leq \tilde{m}|\boldsymbol{\lambda}) \geq Pr(C > \bar{m}|\boldsymbol{\lambda}^*) > 0.$$

($\Leftarrow$): If $\mathbb{P} > 0$, there must exist $\boldsymbol{\lambda}^+$ s.t.

$$w = \int_0^{m^+} Pr(C > t|\boldsymbol{\lambda}^+)dt$$

and $Pr(C > m^+|\boldsymbol{\lambda}^+) < 1$ where $m^+ = \tilde{m}(\boldsymbol{\lambda}^+)$. Since $Pr(C > t|\boldsymbol{\lambda}^+)$ is continuous on $t$,

$$m^+ > \int_0^{m^+} Pr(C > t|\boldsymbol{\lambda}^+)dt = w,$$

so $\bar{m} \geq m^+ > w$.

□

**Lemma 4.** *We have $0 < \underline{p}^{\bar{m}-\epsilon} \leq Pr(C(\boldsymbol{\lambda}) \leq m|\boldsymbol{\lambda}) \leq \bar{p}^{\bar{m}-\epsilon}$ for any $0 < \epsilon < \bar{m} - w$, any $\boldsymbol{\lambda}$ with $\tilde{m}(\boldsymbol{\lambda}) \geq \bar{m} - \epsilon$ and $m \in [\bar{m}-\epsilon, \bar{m}]$, where $\underline{p}^{\bar{m}-\epsilon} := \min_{\boldsymbol{\lambda} \in \Gamma(\epsilon)} Pr(C \leq \bar{m}-\epsilon|\boldsymbol{\lambda})$ and $\bar{p}^{\bar{m}-\epsilon} := \max_{\boldsymbol{\lambda} \in \Gamma(\epsilon)} Pr(C \leq \bar{m}|\boldsymbol{\lambda})$*

*Proof.* According to Lemma 3, there exists $\epsilon$ such that $0 < \epsilon < \bar{m} - w$.

By Assumption 1 and 2, $Pr(C \leq m|\boldsymbol{\lambda})$ is continuous on the compact set $T \times [\bar{m} - \epsilon, \bar{m}]$, so it must have maximum $\bar{p}^{\bar{m}-\epsilon}$ and minimum $\underline{p}^{\bar{m}-\epsilon}$. In addition, $Pr(C(\boldsymbol{\lambda}) \leq m|\boldsymbol{\lambda})$ monotonically increases with $m$. Hence the maximum is reached with $m = \bar{m}$ and the minimum at $m = \bar{m} - \epsilon$.

We only need to prove $\underline{p}^{\bar{m}-\epsilon} > 0$. Assume there exists an $\boldsymbol{\lambda}$ such that $Pr(C \leq \bar{m} - \epsilon|\boldsymbol{\lambda}) = 0$, then by definition,

$$\begin{aligned} w &= \int_0^{\bar{m}} 1 - Pr(C_i \leq t|\boldsymbol{\lambda}_i)dt \\ &\geq \int_o^{\bar{m}-\epsilon} 1 - Pr(C_i \leq t|\boldsymbol{\lambda}_i)dt \\ &\geq \bar{m} - \epsilon \end{aligned}$$

This conflicts with $0 < \epsilon < \bar{m} - w$. Therefore, $\underline{p}^{\bar{m}-\epsilon} > 0$. □

**Lemma 7.** *We have $\lim_{\epsilon \to 0} d(\epsilon) = 0$.*

*Proof.* For any $\boldsymbol{\lambda}$, $\lim_{\epsilon \to 0} Pr(C \leq \bar{m}|\boldsymbol{\lambda}) - Pr(C \leq \bar{m} - \epsilon|\boldsymbol{\lambda}) = 0$ by Assumption 2. In addition, $Pr(C \leq \bar{m}|\boldsymbol{\lambda}) - Pr(C \leq \bar{m} - \epsilon|\boldsymbol{\lambda})$ is continuous and decreasing with respect to $\epsilon$. $\boldsymbol{\lambda}$ is in a compact set. By Dini's Theorem, $Pr(C \leq$

$\bar{m}|\boldsymbol{\lambda}) - Pr(C \leq \bar{m} - \epsilon|\boldsymbol{\lambda})$ absolutely converges to 0, thus giving Lemma 7. □

**Remark 4.** *If the probability distribution function of cost $C$ is locally differentiable in $\Gamma(\epsilon)$, $d(\epsilon)$ can be bounded by:*

$$d(\epsilon) \leq \epsilon \max_{\boldsymbol{\lambda}\in\Gamma(\epsilon), m\in[\bar{m}-\epsilon,\bar{m}]} \frac{d}{dm} Pr(C \leq m|\boldsymbol{\lambda})$$

*This bound may be easier determined in some cases. It is easy to see that this bound goes to 0 too when $\epsilon \to 0$.*

**Theorem 3.** *For any $0 < \epsilon < \bar{m} - w$, we have*

$$R^{-1} \leq \sqrt{\eta_\epsilon + [N/4 + M^2 - \eta_\epsilon] Pr(A)} \quad (7)$$

*where*

$$\begin{aligned} &\eta_\epsilon = K_\epsilon \xi(\bar{p}^{\bar{m}-\epsilon}, \underline{p}^{\bar{m}-\epsilon}) + (K_\epsilon d(\epsilon) + 1/2)^2 \\ &K_\epsilon = \lceil \frac{M - 1/2}{\underline{p}^{\bar{m}-\epsilon}} \rceil \\ &A = \{\tilde{m}_{K_\epsilon+1} < \bar{m} - \epsilon\} \\ &Pr(A) = G(\bar{m}-\epsilon)^N \sum_{i=0}^{K_\epsilon} \binom{N}{i} \left(\frac{1 - G(\bar{m}-\epsilon)}{G(\bar{m}-\epsilon)}\right)^i \\ &G(\bar{m}-\epsilon) = Pr(\tilde{m} < \bar{m} - \epsilon) \end{aligned}$$

*and*

$$\xi(\bar{p}^{\bar{m}-\epsilon}, \underline{p}^{\bar{m}-\epsilon}) = \begin{cases} \underline{p}^{\bar{m}-\epsilon}(1 - \underline{p}^{\bar{m}-\epsilon}) & \text{If } \frac{1}{2} < \underline{p}^{\bar{m}-\epsilon} \\ \bar{p}^{\bar{m}-\epsilon}(1 - \bar{p}^{\bar{m}-\epsilon}) & \text{If } \bar{p}^{\bar{m}-\epsilon} < \frac{1}{2} \\ 1/4, & \text{otherwise} \end{cases}$$

We will provide the proof of Lemma 5, with the help of which we will prove Theorem 3.

**Lemma 5.** *$\forall 0 < \epsilon < \bar{m} - w$, event $A^c$ being $\{\tilde{m}_1 \geq \cdots \geq \tilde{m}_{K_\epsilon+1} \geq \bar{m} - \epsilon\}$, and event $B = \{k \leq K_\epsilon, k'(i) \leq K_\epsilon + 1, \forall i \leq k\}$, we have $A^c \subseteq B$. $Pr(A) = \sum_{i=0}^{K_\epsilon} \binom{N}{i}(1 - G(\bar{m}-\epsilon))^i G(\bar{m}-\epsilon)^{N-i}$. $K_\epsilon = \lceil \frac{M-1/2}{\underline{p}^{\bar{m}-\epsilon}} \rceil$*

*Proof.* By the definition of order statistics, $Pr(A) = \sum_{i=0}^{K_\epsilon} \binom{N}{i}(1 - G(\bar{m}-\epsilon))^i G(\bar{m}-\epsilon)^{N-i}$, so we only need to show $A^c \subseteq B$. From Lemma 4 we have :

$$\sum_{i=1}^{K_\epsilon} Pr(C_i(\boldsymbol{\lambda}_i) \leq \tilde{m}_{K_\epsilon}|\vec{\boldsymbol{\lambda}}) \geq \underline{p}_{\bar{m}-\epsilon} K_\epsilon \geq M - 1/2$$

Since k is the first one to reach $M - 1/2$, the above inequality leads to $k \leq K_\epsilon$.

In addition,

$$\sum_{j=1, j\neq i}^{K_\epsilon} Pr(C_i(\boldsymbol{\lambda}_j) \leq \tilde{m}_{K_\epsilon}|\vec{\boldsymbol{\lambda}}) \geq \underline{p}_{\bar{m}-\epsilon} K_\epsilon \geq M - 1/2$$

so $k'(i) \leq K_\epsilon + 1$ for all $i$. Therefore, $A^c \subseteq B$. □

The proof of Theorem 3 is given as follows.

*Proof.*

$$\text{Error1} \leq M^2 Pr(A) + Pr(A^c) \mathop{\mathbb{E}}_{\vec{\boldsymbol{\lambda}}}[(M - \sum_{i=1}^{k} Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}}))^2 | A^c]$$

noticing that $m_i = \tilde{m}_{k'(i)}$. From Lemma 4

$$\begin{aligned} M + 1/2 &\geq \sum_{i=1}^{k} Pr(C_i(\boldsymbol{\lambda}_i) \leq \tilde{m}_k|\boldsymbol{\lambda}_i) \\ &\geq \sum_{i=1}^{k} Pr(C_i(\boldsymbol{\lambda}_i) \leq \tilde{m}_{k'(i)}|\boldsymbol{\lambda}_i) \\ &\geq \sum_{i=1}^{k} Pr(C_i(\boldsymbol{\lambda}_i) \leq \tilde{m}_k|\boldsymbol{\lambda}_i) - K_\epsilon d(\epsilon) \\ &\geq M - 1/2 - K_\epsilon d(\epsilon) \\ &\Rightarrow \\ &(M - \sum_{i=1}^{k} Pr(C_i(\boldsymbol{\lambda}_i) \leq \tilde{m}_{k'(i)}|\boldsymbol{\lambda}_i))^2 \\ &\leq (K_\epsilon d(\epsilon) + 1/2)^2 \end{aligned}$$

Then the bound of the first term is:

$$\text{Error1} \leq (M^2 - (K_\epsilon d(\epsilon) + 1/2)^2) Pr(A) + (K_\epsilon d(\epsilon) + 1/2)^2 \quad (10)$$

For the second part,

$$\begin{aligned} &\text{Error2} \\ &\leq Pr(A^c) \mathop{\mathbb{E}}_{\vec{\boldsymbol{\lambda}}} \sum_{i=1}^{k} Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})(1 - Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}}))|A^c \\ &+ N/4 Pr(A) \end{aligned}$$

In event $A^c$, $\underline{p}^{\bar{m}-\epsilon} \leq Pr(C(\boldsymbol{\lambda}) \leq \bar{m} - \epsilon|\boldsymbol{\lambda}) \leq \bar{p}^{\bar{m}-\epsilon}$, thus

$$Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})(1 - Pr(C_i \leq m_i|\vec{\boldsymbol{\lambda}})) \leq \xi(\bar{p}^{\bar{m}-\epsilon}, \underline{p}^{\bar{m}-\epsilon})$$

Thus, the bound for the second term will be:

$$\text{Error} \leq K_\epsilon \xi(\bar{p}^{\bar{m}-\epsilon}, \underline{p}^{\bar{m}-\epsilon}) + (N/4 - K_\epsilon \xi(\bar{p}^{\bar{m}-\epsilon}, \underline{p}^{\bar{m}-\epsilon})) Pr(A)$$

Combining the two terms together we shall get the bound for the expected error.

□

**Lemma 6.** *When $\epsilon > 0$,*

$$\lim_{N\to\infty} [N/4 + M^2 - \eta_\epsilon] Pr(A) = 0 \quad (8)$$

*Proof.*

$$Pr(A) = G(\bar{m}-\epsilon)^N \sum_{i=0}^{K_\epsilon} \binom{N}{i} \left(\frac{1 - G(\bar{m}-\epsilon)}{G(\bar{m}-\epsilon)}\right)^i$$

Since $\epsilon > 0$, $G(\bar{m}-\epsilon) < 1$, so $G(\bar{m}-\epsilon)^N$ goes to zero exponentially. The other part $[N/4 + M^2 - \eta_\epsilon] \sum_{i=0}^{K_\epsilon} \binom{N}{i} \left(\frac{1-G(\bar{m}-\epsilon)}{G(\bar{m}-\epsilon)}\right)^i$ only grows polynomially, so the whole term goes to 0 at a fast speed. □

**Corollary 1.** *When the number of agents is infinite,*

$$\lim_{N\to+\infty} R^{-1} \leq \sqrt{(\frac{M-1/2}{\underline{p}^{\bar{m}}}+1)\xi(\bar{p}^{\bar{m}},\underline{p}^{\bar{m}})+1/4} \quad (9)$$

*Proof.* For any fixed $\epsilon \in [0, \bar{m}-w]$, let $N \to \infty$ on both sides of bound 7, according to Lemma 6.

$$\begin{aligned}\lim_{N\to\infty} R^{-1} &\leq \min(\lim_{N\to\infty}\sqrt{\eta_\epsilon+[N/4+M^2-\eta_\epsilon]Pr(A)}, M)\\ &= \min(\sqrt{\eta_\epsilon}, M)\end{aligned}$$

Let $\epsilon \to 0$, since $\lim_{\epsilon\to 0}\bar{p}^{\bar{m}-\epsilon} = \bar{p}^{\bar{m}}, \lim_{\epsilon\to 0}\underline{p}^{\bar{m}-\epsilon} = \underline{p}^{\bar{m}}$, we have $\xi(\bar{p}^{\bar{m}-\epsilon},\underline{p}^{\bar{m}-\epsilon}) \to \xi(\bar{p}^{\bar{m}},\underline{p}^{\bar{m}})$. In addition, $\underline{p}^{\bar{\bar{m}}-\epsilon}$ monotonically decreases with $\epsilon$, and $K_\epsilon$ is left-continuous, so $\lim_{\epsilon\to 0} K_\epsilon \leq K_0+1$. Together with Lemma 7, we have $\eta_\epsilon \to \eta_0$. □