Supplementary Note for 'Distributed Optimal Steady-state Control Using Reverse- and Forward-engineering'

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Detailed proof of Lemma 1: Denote the set of saddle points of f by $\mathcal{S}$. Given any $(y^*, z^*) \in \mathcal{S}$, define a candidate Lyapunov function as

$$V_{(4)} = \frac{1}{2}(y-y^*)^T K_y^{-1}(y-y^*) + \frac{1}{2}(z-z^*)^T K_z^{-1}(z-z^*)$$

which is radially unbounded and positive definite with respect to (y^*, z^*). The derivative of $V_{(4)}$ with respect to time along the trajectory of system (4) in the paper is given by

$$\begin{aligned}\dot{V}_{(4)} =& -\left(\frac{\partial f}{\partial y}\right)^T (y-y^*) + \left(\frac{\partial f}{\partial z}\right)^T (z-z^*)\\ \leq& - f(y,z) + f(y^*,z) + f(y,z) - f(y,z^*)\\ =& f(y^*,z) - f(y^*,z^*) + f(y^*,z^*) - f(y,z^*)\\ \leq& 0\end{aligned}$$

where the first inequality comes from the fact that f is convex in y and concave in z, and the last inequality follows that (y^*, z^*) is a saddle point of f. When $\dot{V}_{(4)} = 0$, we have $f(y^*,z) = f(y^*,z^*)$ and $f(y^*,z^*) = f(y,z^*)$. Based on LaSalle's invariance principle [1], we conclude that the trajectories of (4) converge to a compact subset of the invariant set given by

$$\mathcal{I} = \{(y,z) | \dot{V}_{(4)}(y,z,y^*,z^*) = 0\}$$

which indicates that the trajectories of (4) are bounded.

We next show that if f is either strictly convex in y or strictly concave in z, then each trajectory of (4) asymptotically converges to a saddle point of f. Firstly, if $(\tilde{y}, \tilde{z}) \in \mathcal{I}$, then $f(y^*, \tilde{z}) = f(y^*, z^*)$ and $f(y^*, z^*) = f(\tilde{y}, z^*)$. Since f is either strictly convex in y or strictly concave in z, either $\tilde{y} = y^*$ or $\tilde{z} = z^*$ holds. Let us assume $\tilde{y} = y^*$, which means $\dot{y}|_{y=\tilde{y}} = \mathbf{0}$. Then $\frac{\partial f}{\partial y}|_{y=\tilde{y}} = \mathbf{0}$, i.e., $\tilde{y}$ satisfies the first order optimality condition for $f(y,z)$. On the other hand, since $f(\tilde{y}, z^*) = f(y^*, z^*) = f(y^*, \tilde{z})$, $\tilde{z}$ is a maximizer of $f(y^*, z)$, i.e., $\frac{\partial f}{\partial z}|_{y=y^*=\tilde{y}, z=\tilde{z}} = \mathbf{0}$. Thus, we have $\nabla_{y,z} f|_{y=\tilde{y}, z=\tilde{z}} = \mathbf{0}$, i.e., $(\tilde{y}, \tilde{z}) \in \mathcal{S}$. Similarly, if $\tilde{z} = z^*$, we can still derive $(\tilde{y}, \tilde{z}) \in \mathcal{S}$. To conclude, under the condition that f is either strictly convex in y or strictly concave in z, $\mathcal{I} \subseteq \mathcal{S}$ is true.

To show the pointwise convergence, since $(y(t), z(t))$ converges to a compact subset of $\mathcal{I} \subseteq \mathcal{S}$, there exists a subsequence $\{(y_k, z_k)\}$ where $y_k = y(t_k), z_k = z(t_k)$ that converges to a point (y^∞, z^∞). This means that $v_k = V_{(4)}(y_k, z_k, y^\infty, z^\infty)$ asymptotically converges to 0. On the other hand, we have

$$\lim_{t\to\infty} V_{(4)}(y(t), z(t), y^\infty, z^\infty) = \lim_{t\to\infty} v(t) = v^\infty$$

where v^∞ is constant. Since v_k is a subsequence of $v(t)$ and converges to 0, $v^\infty = 0$ holds. Therefore, we conclude that $(y(t), z(t))$ converges to (y^∞, z^∞). ∎

Detailed proof of Lemma 2: Due to $\nabla^2_{y,u,x^{(2)}} L_{\text{au}} = \nabla^2_{y,u,x^{(2)}} L_{\text{sys}} + \gamma \nabla^2_{y,u,x^{(2)}} L_{\text{op}}$, $\nabla^2_{y,u,x^{(2)}} L_{\text{sys}} \succeq 0$ and $\nabla^2_{y,u,x^{(2)}} L_{\text{op}} \succeq 0$, we have $\nabla^2_{y,u,x^{(2)}} L_{\text{au}} \succeq 0$. Similarly, $\nabla^2_{\zeta_i,\lambda_i,\mu_i,x^{(1)}} L_{\text{au}} \preceq 0$ holds. Based on the KKT conditions [2] (saddle points of a function satisfy the KKT conditions), any saddle point of L_{au} satisfies

$$\sum_{j\in\mathcal{N}(i)} A_{ij}x_j + B_iu_i + C_iw_i = \mathbf{0} \tag{1a}$$

$$\sum_{j\in\mathcal{N}(i)} D_{ij}x_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_iw_i - \gamma P_{u_i}\left(\frac{\partial g_i}{\partial u_i} - B_i^T\zeta_i - \sum_{j\in\mathcal{N}(i)} E_{ji}^T\lambda_j + \mu_i\frac{\partial h_i}{\partial u_i}\right) = \mathbf{0} \tag{1b}$$

$$\frac{\partial f_i}{\partial y_i} - \sum_{j\in\mathcal{N}(i)} A_{ji}^T\zeta_j - \sum_{j\in\mathcal{N}(i)} D_{ji}^T\lambda_j + \mu_i\frac{\partial h_i}{\partial y_i} = \mathbf{0} \tag{1c}$$

$$\sum_{j\in\mathcal{N}(i)} A_{ij}y_j + B_iu_i + C_iw_i = \mathbf{0} \tag{1d}$$

$$\sum_{j\in\mathcal{N}(i)} D_{ij}y_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_iw_i = \mathbf{0} \tag{1e}$$

$$\mu_ih_i(y_i,u_i) = 0, \;\; \mu_i \geq 0, \;\; h_i(y_i,u_i) \leq 0 \tag{1f}$$

where $i = 1, \cdots, N$. If A is invertible in system (1) in the paper, from (1a), (1d)-(1e) in the above equations, $y = x$ and $\sum_{j\in\mathcal{N}(i)} D_{ij}x_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_iw_i = \mathbf{0}$ hold. On the other hand, any saddle point of L_{sys} satisfies

$$\sum_{j\in\mathcal{N}(i)} A_{ij}x_j + B_iu_i + C_iw_i = \mathbf{0}$$
$$\sum_{j\in\mathcal{N}(i)} D_{ij}x_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_iw_i = \mathbf{0}$$

where $i = 1, \cdots, N$, and any saddle point of L_{op} satisfies

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$$\frac{\partial g_i}{\partial u_i} - B_i^T\zeta_i - \sum_{j\in\mathcal{N}(i)} E_{ji}^T\lambda_j + \mu_i\frac{\partial h_i}{\partial u_i} = \mathbf{0}$$
$$\frac{\partial f_i}{\partial y_i} - \sum_{j\in\mathcal{N}(i)} A_{ji}^T\zeta_j - \sum_{j\in\mathcal{N}(i)} D_{ji}^T\lambda_j + \mu_i\frac{\partial h_i}{\partial y_i} = \mathbf{0}$$
$$\sum_{j\in\mathcal{N}(i)} A_{ij}y_j + B_iu_i + C_iw_i = \mathbf{0}$$
$$\sum_{j\in\mathcal{N}(i)} D_{ij}y_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_iw_i = \mathbf{0}$$
$$\mu_ih_i(y_i,u_i) = 0,\ \ \mu_i \geq 0,\ \ h_i(y_i,u_i) \leq 0$$

where $i = 1,\cdots,N$. Therefore, the following two sets are equivalent:

$\{(y,u,x,\zeta_i,\lambda_i,\mu_i)|(y,u,x,\zeta_i,\lambda_i,\mu_i)$ is a saddle point of $L_{\text{au}}\} \Leftrightarrow \{(y,u,x,\zeta_i,\lambda_i,\mu_i)|(x,u)$ is a saddle point of L_{sys}, $(y,u,\zeta_i,\lambda_i,\mu_i)$ is a saddle point of $L_{\text{op}}\}$

and furthermore, $y = x$ holds. Thus, $(y,u,x,\zeta_i,\lambda_i,\mu_i)$ is a saddle point of L_{au} if and only if $(y,u,\zeta_i,\lambda_i,\mu_i)$ is a saddle point of L_{op} and (x,u) is a saddle point of L_{sys}. ■

Detailed proof of Lemma 3: Let (y^*,z^*) be a saddle point of f. Define a candidate Lyapunov function for system (9) in the paper as

$$V_{(9)} = \frac{1}{2}\Big((y-y^*)^TK_y^{-1}(y-y^*) + (z-z^*)^TK_z^{-1}(z-z^*) + (\hat{y}-y^*)^TK_{ey}\hat{K}_{ey}^{-1}(\hat{y}-y^*) + (\hat{z}-z^*)^TK_{ez}\hat{K}_{ez}^{-1}(\hat{z}-z^*)\Big)$$

which is radially unbounded and positive definite with respect to (y^*,z^*,y^*,z^*) (note that at any equilibrium of (9), $y^* = \hat{y}^*, z^* = \hat{z}^*$ hold). The derivative of $V_{(9)}$ with respect to time along the trajectory of system (9) is given by

$$\begin{aligned}\dot{V}_{(9)} =& -\left(\frac{\partial f}{\partial y}\right)^T(y-y^*) + \left(\frac{\partial f}{\partial z}\right)^T(z-z^*)\\ &-(y-\hat{y})^TK_{ey}(y-\hat{y}) - (z-\hat{z})^TK_{ez}(z-\hat{z})\\ \leq& -f(y,z) + f(y^*,z) + f(y,z) - f(y,z^*)\\ &-(y-\hat{y})^TK_{ey}(y-\hat{y}) - (z-\hat{z})^TK_{ez}(z-\hat{z})\\ =& f(y^*,z) - f(y^*,z^*) + f(y^*,z^*) - f(y,z^*)\\ &-(y-\hat{y})^TK_{ey}(y-\hat{y}) - (z-\hat{z})^TK_{ez}(z-\hat{z})\\ \leq& 0\end{aligned}$$

where the first inequality comes from the fact that f is convex in y and concave in z, and the last inequality follows that (y^*,z^*) is a saddle point of f. When $\dot{V}_{(9)} = 0$, we have $y = \hat{y}, z = \hat{z}$, leading to $\dot{\hat{y}} = \dot{y} = \mathbf{0}, \dot{\hat{z}} = \dot{z} = \mathbf{0}$, $\frac{\partial f}{\partial y} = \mathbf{0}$ and $\frac{\partial f}{\partial z} = \mathbf{0}$. From LaSalle's invariance principle [1], we conclude that each trajectory of system (9) converges to a compact subset of the invariant set

$$\mathcal{I} = \{(y,z,\hat{y},\hat{z})|\dot{V}_{(9)}(y,z,\hat{y},\hat{z},y^*,z^*) = 0\}$$

which is a subset of

$\mathcal{S} = \{(y,z,\hat{y},\hat{z})|(y,z)$ is a saddle point of $f, \hat{y} = y, \hat{z} = z\}$.

Following the proof of Lemma 1, pointwise convergence holds. Thus, we conclude that each trajectory of system (9) asymptotically converges to an equilibrium point at which (y,z) is a saddle point of f. ■

Detailed proof of Theorem 1: Since A is Hurwitz in system (1) in the paper, thus invertible, based on Lemma 2, at any equilibrium point of system (8) in the paper which is also a saddle point of L_{au}, (x,u) ($x = y$ holds) is an optimal solution of problem (3) in the paper. We next prove convergence. Let $(y^*,u^*,x^*,\zeta_i^*,\lambda_i^*,\mu_i^*)$ be a saddle point of L_{au} where $\mu_i^* \geq 0$. Define a candidate Lyapunov function as

$$\begin{aligned}V_{(8)} =& \frac{1}{2}\sum_{i=1}^N\Big(\gamma(y_i-y_i^*)^TK_{y_i}^{-1}(y_i-y_i^*) + \gamma(\hat{y}_i-y_i^*)^TK_{ey_i}\\ &\times\hat{K}_{ey_i}^{-1}(\hat{y}_i-y_i^*) + (u_i-u_i^*)^TP_{u_i}^{-1}(u_i-u_i^*) + \gamma(\hat{u}_i-u_i^*)^T\\ &\times K_{eu_i}\hat{K}_{eu_i}^{-1}(\hat{u}_i-u_i^*) + \gamma(\zeta_i-\zeta_i^*)^TK_{\zeta_i}^{-1}(\zeta_i-\zeta_i^*) + \gamma(\hat{\zeta}_i\\ &-\zeta_i^*)^TK_{e\zeta_i}\hat{K}_{e\zeta_i}^{-1}(\hat{\zeta}_i-\zeta_i^*) + \gamma(\lambda_i-\lambda_i^*)^TK_{\lambda_i}^{-1}(\lambda_i-\lambda_i^*)\\ &+\gamma(\hat{\lambda}_i-\lambda_i^*)^TK_{e\lambda_i}\hat{K}_{e\lambda_i}^{-1}(\hat{\lambda}_i-\lambda_i^*) + \gamma k_{\mu_i}^{-1}(\mu_i-\mu_i^*)^2\Big)\\ &+\frac{1}{2}(x-x^*)^T\text{diag}\{P_{x^{(1)}},P_{x^{(2)}}\}^{-1}(x-x^*)\end{aligned}$$

which is positive definite with respect to $(y^*,u^*,x^*,\zeta_i^*,\lambda_i^*,\mu_i^*,y^*,u^*,\zeta_i^*,\lambda_i^*)$ (note that at any equilibrium of (8), $y^* = \hat{y}^*, u^* = \hat{u}^*, \zeta_i^* = \hat{\zeta}_i^*, \lambda_i^* = \hat{\lambda}_i^*$ hold). The derivative of $V_{(8)}$ with respect to time along the trajectory of (8) is given by

$$\begin{aligned}\dot{V}_{(8)} =& \sum_{i=1}^N\Bigg(-\left(\frac{\partial L_{\text{au}}}{\partial y_i}\right)^T(y_i-y_i^*) - \left(\frac{\partial L_{\text{au}}}{\partial u_i}\right)^T(u_i-u_i^*)\\ &+\left(\frac{\partial L_{\text{au}}}{\partial \zeta_i}\right)^T(\zeta_i-\zeta_i^*) + \left(\frac{\partial L_{\text{au}}}{\partial \lambda_i}\right)^T(\lambda_i-\lambda_i^*) + \gamma(\mu_i-\mu_i^*)\\ &\times(h_i(y_i,u_i))_{\mu_i}^+ - \gamma(y_i-\hat{y}_i)^TK_{ey_i}(y_i-\hat{y}_i) - \gamma(u_i-\hat{u}_i)^T\\ &\times K_{eu_i}(u_i-\hat{u}_i) - \gamma(\zeta_i-\hat{\zeta}_i)^TK_{e\zeta_i}(\zeta_i-\hat{\zeta}_i) - \gamma(\lambda_i-\hat{\lambda}_i)^T\\ &\times K_{e\lambda_i}(\lambda_i-\hat{\lambda}_i)\Bigg) + \left(\frac{\partial L_{\text{au}}}{\partial x^{(1)}}\right)^T(x^{(1)}-x^{(1)*}) - \left(\frac{\partial L_{\text{au}}}{\partial x^{(2)}}\right)^T\\ &\times(x^{(2)}-x^{(2)*}).\end{aligned}$$

Note that

$$\begin{aligned}\gamma(\mu_i-\mu_i^*)(h_i(y_i,u_i))_{\mu_i}^+ \leq& \gamma(\mu_i-\mu_i^*)h_i(y_i,u_i)\\ =& \left(\frac{\partial L_{\text{au}}}{\partial \mu_i}\right)^T(\mu_i-\mu_i^*)\end{aligned}$$

because when $(h_i(y_i,u_i))_{\mu_i}^+$ is inactive, $h_i(y_i,u_i)$ is non-positive (otherwise, it is positive and then $(h_i(y_i,u_i))_{\mu_i}^+$ would be active which indicates the equality case), and $(\mu_i - \mu_i^*)$ is also non-positive so that $(\mu_i-\mu_i^*)(h_i(y_i,u_i))_{\mu_i}^+ \leq (\mu_i-\mu_i^*)h_i(y_i,u_i)$ is always true. Therefore, we have

$$\begin{aligned}\dot{V}_{(8)} \leq& \sum_{i=1}^N\Bigg(-\left(\frac{\partial L_{\text{au}}}{\partial y_i}\right)^T(y_i-y_i^*) - \left(\frac{\partial L_{\text{au}}}{\partial u_i}\right)^T(u_i-u_i^*)\\ &+\left(\frac{\partial L_{\text{au}}}{\partial \zeta_i}\right)^T(\zeta_i-\zeta_i^*) + \left(\frac{\partial L_{\text{au}}}{\partial \lambda_i}\right)^T(\lambda_i-\lambda_i^*) + \left(\frac{\partial L_{\text{au}}}{\partial \mu_i}\right)^T\\ &\times(\mu_i-\mu_i^*) - \gamma(y_i-\hat{y}_i)^TK_{ey_i}(y_i-\hat{y}_i) - \gamma(u_i-\hat{u}_i)^T\end{aligned}$$

$\times K_{eu_i}(u_i - \hat{u}_i) - \gamma(\zeta_i - \hat{\zeta}_i)^T K_{e\zeta_i}(\zeta_i - \hat{\zeta}_i) - \gamma(\lambda_i - \hat{\lambda}_i)^T$
$\times K_{e\lambda_i}(\lambda_i - \hat{\lambda}_i)\Big) + \left(\frac{\partial L_{\text{au}}}{\partial x^{(1)}}\right)^T (x^{(1)} - x^{(1)*}) - \left(\frac{\partial L_{\text{au}}}{\partial x^{(2)}}\right)^T$
$\times (x^{(2)} - x^{(2)*})$

which has the same structure as $\dot{V}_{(9)}$ in the proof of Lemma 3. So $\dot{V}_{(8)} \le 0$ is true. If $\dot{V}_{(8)} = 0$, $u_i = \hat{u}_i$, $y_i = \hat{y}_i$, $\zeta_i = \hat{\zeta}_i$, $\lambda_i = \hat{\lambda}_i$ hold, which lead to $\dot{\hat{u}}_i = \dot{u}_i = \mathbf{0}, \dot{\hat{y}}_i = \dot{y}_i = \mathbf{0}, \dot{\hat{\zeta}}_i = \dot{\zeta}_i = \mathbf{0}, \dot{\hat{\lambda}}_i = \dot{\lambda}_i = \mathbf{0}$. Under constant u and w, system (1) eventually converges to an equilibrium point at which $x = y$ since A is Hurwitz. In addition, each μ_i is constant due to the fact that $\dot{u}_i = \mathbf{0}, \sum_{j\in\mathcal{N}(i)} D_{ij}x_j + \sum_{j\in\mathcal{N}(i)} E_{ij}u_j + F_i w_i = \mathbf{0}, \dot{y}_i = \mathbf{0}$. So $\dot{\mu}_i = 0, i = 1, \cdots, N$ are true. From LaSalle's invariance principle [1], we conclude that each trajectory of system (8) converges to a compact subset of the invariant set

$$\mathcal{I} =\{(y,u,x,\zeta_i,\lambda_i,\mu_i,\hat{y},\hat{u},\hat{\zeta}_i,\hat{\lambda}_i)|\dot{V}_{(8)}(y,u,x,\zeta_i,\lambda_i,\mu_i,\hat{y}, \hat{u},\hat{\zeta}_i,\hat{\lambda}_i,y^*,u^*,x^*,\zeta_i^*,\lambda_i^*,\mu_i^*) = 0\}$$

which is a subset of

$$\mathcal{S} =\{(y,u,x,\zeta_i,\lambda_i,\mu_i,\hat{y},\hat{u},\hat{\zeta}_i,\hat{\lambda}_i)|(y,u,x,\zeta_i,\lambda_i,\mu_i) \text{ is a saddle point of } L_{\text{au}}, \hat{y} = y, \hat{u} = u, \hat{\zeta}_i = \zeta_i, \hat{\lambda}_i = \lambda_i\}.$$

Finally, pointwise convergence can be shown following the proof of Lemma 1. ■

Note that if the optimization problem (3) in the paper is feasible and satisfies Slater's constraint qualification [2] (this is stated after problem (3) in the paper), the equilibrium set of system (8) in the paper is always nonempty (points in this set satisfy the KKT conditions of problem (3)). The uniqueness of the equilibrium results from the strict convexity of the objective function of problem (3). If it is strictly convex, the optimal solution of (3) is unique. According to Theorem 1, all trajectories of (8) converge to an equilibrium point where (x, u) is the optimal solution of (3). Otherwise, (3) could have multiple optimal solutions and each trajectory of (8) converges to one equilibrium point in the equilibrium set depending on the initial condition.

Detailed proof of Corollary 1: Necessity. Suppose that system (20) in the paper belongs to Class-$\mathcal{S}'$ and is a primal-dual gradient algorithm to solve $\max_{x^{(1)}} \min_{x^{(2)},u} L_{\text{sys}}$,

$$L_{\text{sys}} = \frac{1}{2}\tilde{x}^T \underbrace{\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}}_{Q} \tilde{x} + \tilde{x}^T P^{-1}\tilde{C}w$$

where $Q_{11} \in \mathbb{R}^{n_1\times n_1}$ satisfies $Q_{11} = Q_{11}^T \preceq 0$ (i.e., L_{sys} is concave in $x^{(1)}$), $Q_{22} \in \mathbb{R}^{(n_2+m)\times(n_2+m)}$ satisfies $Q_{22} = Q_{22}^T \succeq 0$ (i.e., L_{sys} is convex in $(x^{(2)}, u)$), $Q_{12} \in \mathbb{R}^{n_1\times(n_2+m)}$ and $P \in \mathbb{R}^{(n_1+n_2+m)\times(n_1+n_2+m)}$. Then the trajectories of (20) are bounded under Lemma 1, and there exist matrices $P_{x^{(1)}} \in \mathbb{R}^{n_1\times n_1}$, $P_{x^{(2)}} \in \mathbb{R}^{n_2\times n_2}$ and $P_{u_i} \in \mathbb{R}^{m_i\times m_i}$ satisfying $P_{x^{(1)}} = P_{x^{(1)}}^T \succ 0$, $P_{x^{(2)}} = P_{x^{(2)}}^T \succ 0$ and $P_{u_i} = P_{u_i}^T \succ 0, i = 1, \cdots, N$, so that the equality $Q = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}, -\text{diag}\{P_{u_i}\}\}^{-1}A = A^T\text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}, -\text{diag}\{P_{u_i}\}\}^{-1}$ holds (also let $P = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}, -\text{diag}\{P_{u_i}\}\}$). This leads to

$$P_{x^{(1)}}^{-1}A_{11} = A_{11}^T P_{x^{(1)}}^{-1} = Q_{11} \preceq 0$$
$$\begin{bmatrix} P_{x^{(2)}}^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{P_{u_i}^{-1}\} \end{bmatrix}\begin{bmatrix} A_{22} & B_2 \\ D_2 & E \end{bmatrix} = \begin{bmatrix} A_{22}^T & D_2^T \\ B_2^T & E^T \end{bmatrix} \times \begin{bmatrix} P_{x^{(2)}}^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{P_{u_i}^{-1}\} \end{bmatrix} = -Q_{22} \preceq 0$$
$$P_{x^{(1)}}^{-1}\begin{bmatrix} A_{12} & B_1 \end{bmatrix} = -\begin{bmatrix} A_{21}^T & D_1^T \end{bmatrix}\begin{bmatrix} P_{x^{(2)}}^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{P_{u_i}^{-1}\} \end{bmatrix}$$

which can be further rearranged as

$$P_{x^{(1)}}^{-1}A_{11} = A_{11}^T P_{x^{(1)}}^{-1} \preceq 0$$
$$P_{x^{(2)}}^{-1}A_{22} = A_{22}^T P_{x^{(2)}}^{-1} \preceq 0$$
$$P_{x^{(2)}}^{-1}B_2 = D_2^T\text{diag}\{P_{u_i}^{-1}\}$$
$$\text{diag}\{P_{u_i}^{-1}\}\begin{bmatrix} E_{11} & \cdots & E_{1N} \\ \cdots & \cdots & \cdots \\ E_{N1} & \cdots & E_{NN} \end{bmatrix} = \begin{bmatrix} E_{11}^T & \cdots & E_{N1}^T \\ \cdots & \cdots & \cdots \\ E_{1N}^T & \cdots & E_{NN}^T \end{bmatrix} \times \text{diag}\{P_{u_i}^{-1}\} \preceq 0$$
$$P_{x^{(1)}}^{-1}A_{12} + A_{21}^T P_{x^{(2)}}^{-1} = \mathbf{0}$$
$$P_{x^{(1)}}^{-1}B_1 + D_1^T\text{diag}\{P_{u_i}^{-1}\} = \mathbf{0}.$$

Based on Lemma 4, the above equations are equivalent to conditions (ii) and (iii) by defining $V_1 = J_1^T P_{x^{(1)}}^{-1}J_1, V_2 = J_2^T P_{x^{(2)}}^{-1}J_2, V_{E_i} = J_{E_i}^T P_{u_i}^{-1}J_{E_i}, i = 1, \cdots, N$.

Sufficiency. Let conditions (i)-(iii) be true. Consider the following unconstrained quadratic saddle point problem:

$$\max_{x^{(1)}\in\mathbb{R}^{n_1}} \min_{x^{(2)}\in\mathbb{R}^{n_2}, u\in\mathbb{R}^m} L_{\text{sys}} = \frac{1}{2}\tilde{x}^T P^{-1}\tilde{A}\tilde{x} + \tilde{x}^T P^{-1}\tilde{C}w$$

where $P^{-1} = \text{diag}\{(J_1^{-1})^T V_1 J_1^{-1}, -(J_2^{-1})^T V_2 J_2^{-1}, -\text{diag}\{(J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\}\}$. Due to $V_1\Lambda_1 \preceq 0$ and

$$\begin{bmatrix} (J_2^{-1})^T V_2 J_2^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\} \end{bmatrix}\begin{bmatrix} A_{22} & B_2 \\ D_2 & E \end{bmatrix} = \begin{bmatrix} A_{22}^T & D_2^T \\ B_2^T & E^T \end{bmatrix}\begin{bmatrix} (J_2^{-1})^T V_2 J_2^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\} \end{bmatrix} \preceq 0 \quad (2)$$

L_{sys} is concave in $x^{(1)}$ and convex in $(x^{(2)}, u)$. Equation (2) is derived as follow. Let $\tilde{A}_{22} = \begin{bmatrix} A_{22} & B_2 \\ D_2 & E \end{bmatrix}$ and $\tilde{P}_2^{-1} = \begin{bmatrix} (J_2^{-1})^T V_2 J_2^{-1} & \mathbf{0} \\ \mathbf{0} & \text{diag}\{J_{E_i}^{-1})^T V_{E_i} J_{E_i}^{-1}\} \end{bmatrix}$. Since the eigenvalues of $\tilde{A}_{22}$ are non-positive real and $\tilde{A}_{22}$ is diagonalizable, $\tilde{A}_{22}$ can be written as $\tilde{A}_{22} = \tilde{J}\tilde{\Lambda}\tilde{J}^{-1}$ where $\tilde{\Lambda} \preceq 0$ is a diagonal matrix. Due to $\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda}\tilde{J}^{-1} = (\tilde{J}^{-1})^T\tilde{\Lambda}\tilde{J}^T\tilde{P}_2^{-1}$, based on Lemma 1 in [3], $\tilde{J}^T\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda} = \tilde{\Lambda}\tilde{J}^T\tilde{P}_2^{-1}\tilde{J} \preceq 0$ holds. For any given vector $v \in \mathbb{R}^{n_2+m}$, $(\tilde{J}v)^T\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda}\tilde{J}^{-1}(\tilde{J}v) = v^T\tilde{J}^T\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda}v \le 0$, which leads to $\tilde{P}_2^{-1}\tilde{J}\tilde{\Lambda}\tilde{J}^{-1} \preceq 0$, i.e., Equation (2) holds.

Now define matrices $P_{x^{(1)}} = J_1 V_1^{-1} J_1^T \succ 0$, $P_{x^{(2)}} = J_2 V_2^{-1} J_2^T \succ 0$ and $P_{u_i} = J_{E_i} V_{E_i}^{-1} J_{E_i}^T \succ 0, i = 1, \cdots, N$. Under Lemma 1, the trajectories of the primal-dual gradient algorithm given by $\dot{x} = \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}\}\frac{\partial L_{\text{sys}}}{\partial x}$ and

$\dot{u}_i = -P_{u_i}\frac{\partial L_{\text{sys}}}{\partial u_i}, i = 1, \cdots, N$ are bounded, which is the same as (20). So we conclude that system (20) belongs to Class-$\mathcal{S}'$. ■

Detailed case study: Here we provide some details of the practical example given in Section V in the paper. The network configuration is presented in [4]. For conciseness, we directly present the system model with a decentralized integral controller in state-space form:

Power network dynamics:

$$M\dot{\omega}_g + D_g\omega_g = P_M - d_g - \Gamma_1 ATA^T\alpha \quad (3a)$$

$$\dot{\alpha} = \Gamma_1^T\omega_g + \Gamma_2^T\omega_l \quad (3b)$$

$$\omega_l = D_l^{-1}(-P_L - d_l - \Gamma_2 ATA^T\alpha) \quad (3c)$$

$$\dot{P}_M = T_{TG}^{-1}(P_C - P_M - R^{-1}\omega_g) \quad (3d)$$

Control input dynamics:

$$\dot{P}_C = K_{P_C}(R(P_M - P_C)) \quad (3e)$$

$$\dot{P}_L = K_{P_L}\omega_l \quad (3f)$$

where $\alpha = [\alpha_1, \cdots, \alpha_{|\mathcal{G}\bigcup\mathcal{L}|-1}]^T$, $\alpha_i = \delta_i - \delta_{|\mathcal{G}\bigcup\mathcal{L}|}$, i.e., one of the buses with number $|\mathcal{G}\bigcup\mathcal{L}|$ is regarded as a reference bus, and K_{P_C}, K_{P_L} are positive diagonal matrices representing the controller gains.

The optimization problem corresponding to problem (3) in the paper is:

$$\min_{P_M, P_L, \alpha} \sum_{i\in\mathcal{G}} C_i(P_{M_i}) - \sum_{i\in\mathcal{L}} U_i(P_{L_i}) \quad (4a)$$

$$\text{subject to } P_M - d_g - \Gamma_1 ATA^T\alpha = \mathbf{0} \quad (4b)$$

$$-P_L - d_l - \Gamma_2 ATA^T\alpha = \mathbf{0} \quad (4c)$$

$$P_C^{\min} \preceq P_M \preceq P_C^{\max} \quad (4d)$$

$$P_L^{\min} \preceq P_L \preceq P_L^{\max} \quad (4e)$$

$$P_{TC}^{\min} \preceq TA^T\alpha \preceq P_{TC}^{\max} \quad (4f)$$

where $C_i(P_{M_i})$ is the cost function for each generator and is a strictly convex function in P_{M_i}, $U_i(P_{L_i})$ is the utility function for each controllable load and is a strictly concave function in P_{L_i}, Equations (4b)-(4c) represent power flow balance at each bus, Equations (4d)-(4f) are capacity constraints for generators, loads, and transmission lines respectively, and $P_C^{\min}, P_C^{\max}, P_L^{\min}, P_L^{\max}, P_{TC}^{\min}, P_{TC}^{\max}$ are corresponding capacity vectors. Note that by setting $P_{TC_{ij}}^{\max} = P_{TC_{ij}}^{\min}$, the scheduled power flow in line (i, j) can be maintained.

Since the dynamics of P_M and α are constrained by (3) during the transient, they therefore cannot be instantaneously set to the solution of problem (4). Note that Equations (4b)-(4c) require the frequency to be restored when system (3) reaches steady state, i.e., $\omega_i^* = 0, i \in \mathcal{G}\bigcup\mathcal{L}$. This results in $P_M = P_C$ in steady state. So we can reformulate problem (4) as one that can be used to design P_C and P_L, given by

$$\min_{P_C, P_L, \theta} \sum_{i\in\mathcal{G}} C_i(P_{C_i}) - \sum_{i\in\mathcal{L}} U_i(P_{L_i}) \quad (5a)$$

$$\text{subject to } P_C - d_g - \Gamma_1 ATA_0^T\theta = \mathbf{0} \quad (5b)$$

$$-P_L - d_l - \Gamma_2 ATA_0^T\theta = \mathbf{0} \quad (5c)$$

$$P_C^{\min} \preceq P_C \preceq P_C^{\max} \quad (5d)$$

$$P_L^{\min} \preceq P_L \preceq P_L^{\max} \quad (5e)$$

$$P_{TC}^{\min} \preceq TA_0^T\theta \preceq P_{TC}^{\max} \quad (5f)$$

where $\theta = [\theta_1, \cdots, \theta_{m+n}]^T$ is the vector of ancillary decision variables. In fact, we have replaced P_M, α in (4) by P_C, θ to derive (5).

It can be shown that the optimality of problem (4) is preserved after the reformulation. Finally, the real-time economic dispatch problem is described as: design P_C and P_L so that for given constant d_g, d_l, system (3) is driven to an equilibrium point where the steady-state optimization problem (4)/(5) is solved.

Using Corollary 1, we can reverse-engineer system (3) as one with primal-dual gradient dynamics to solve a saddle point problem

$$\min_{\alpha, P_M, P_C, P_L} \max_{\omega_g} L_{(3)} \quad (6)$$

and $L_{(3)}$ is given by

$$L_{(3)} = -\frac{1}{2}(D_g\omega_g - P_M + d_g + \Gamma_1 ATA^T\alpha)^T D_g^{-1}(D_g\omega_g - P_M + d_g + \Gamma_1 ATA^T\alpha) + \frac{1}{2}(P_M - d_g - \Gamma_1 ATA^T\alpha)^T D_g^{-1} \times (P_M - d_g - \Gamma_1 ATA^T\alpha) + \frac{1}{2}(P_L + d_l + \Gamma_2 ATA^T\alpha)^T D_l^{-1} \times (P_L + d_l + \Gamma_2 ATA^T\alpha) + \frac{1}{2}(P_M - P_C)^T R(P_M - P_C)$$

where $\alpha \in \mathbb{R}^{|\mathcal{G}\bigcup\mathcal{L}|-1}, P_M \in \mathbb{R}^{|\mathcal{G}|}, \omega_g \in \mathbb{R}^{|\mathcal{G}|}, P_C \in \mathbb{R}^{|\mathcal{G}|}, P_L \in \mathbb{R}^{|\mathcal{L}|}$ are decision variables, and $d_g \in \mathbb{R}^{|\mathcal{G}|}, d_l \in \mathbb{R}^{|\mathcal{L}|}$ are constant. Note that for simplicity, we have used the same notation as in the dynamics (3). Through straightforward derivation, we can show that system dynamics (3) can be rewritten as

$$\frac{\partial L_{(3)}}{\partial \omega_g} = M\dot{\omega}_g \ (\dot{\omega}_g \text{ is given in (3a)})$$

$$\frac{\partial L_{(3)}}{\partial \alpha} = -ATA^T\dot{\alpha} \ (\ \dot{\alpha} \text{ is given in (3b)})$$

$$\frac{\partial L_{(3)}}{\partial P_M} = -T_{TG}R\dot{P}_M \ (\dot{P}_M \text{ is given in (3d)})$$

$$\frac{\partial L_{(3)}}{\partial P_C} = -K_{P_C}^{-1}\dot{P}_C \ (\dot{P}_C \text{ is given in (3e)})$$

$$\frac{\partial L_{(3)}}{\partial P_L} = -K_{P_L}^{-1}\dot{P}_L \ (\dot{P}_L \text{ is given in (3f)}).$$

Now for problem (5), formulate its Lagrangian as

$$L_{(5)}(P_C, P_L, \theta, \zeta, \lambda, \mu^+, \mu^-, \nu^+, \nu^-, l^+, l^-) = \sum_{i\in\mathcal{G}} C_i(P_{C_i}) - \sum_{i\in\mathcal{L}} U_i(P_{L_i}) + \zeta^T(P_C - d_g - \Gamma_1 ATA_0^T\theta) + \lambda^T(-P_L - d_l - \Gamma_2 ATA_0^T\theta) + \mu^{+T}(P_C - P_C^{\max}) + \mu^{-T}(P_C^{\min} - P_C) + \nu^{+T} \times (P_L - P_L^{\max}) + \nu^{-T}(P_L^{\min} - P_L) + l^{+T}(TA_0^T\theta - P_{TC}^{\max}) + l^{-T}(P_{TC}^{\min} - TA_0^T\theta)$$

where $\zeta, \mu^+, \mu^-, \lambda, \nu^+, \nu^-, l^+, l^-$ are Lagrange multipliers (dual vectors) for the constraints in (5). Then we obtain the following saddle point problem:

$$\min_{P_C, P_L, \theta} \max_{\mu^+, \mu^-, \nu^+, \nu^-, l^+, l^- \succeq 0, \zeta, \lambda} L_{(5)}. \tag{7}$$

As a result, solving problem (5) is equivalent to solving problem (7). Consider the augmented saddle point problem given by

$$\min_{\alpha, P_M, P_C, P_L, \theta} \max_{\mu^+, \mu^-, \nu^+, \nu^-, l^+, l^- \succeq 0, \zeta, \lambda, \omega_g} L_{\text{au}} = L_{(3)} + \gamma L_{(5)} \tag{8}$$

where $\gamma > 0$ is constant. A distributed dynamic feedback controller to solve the real-time economic dispatch problem can then be derived using L_{au} and Lemma 1, given by

Power network dynamics:

$$M\dot{\omega}_g + D_g \omega_g = P_M - d_g - \Gamma_1 A T A^T \alpha \tag{9a}$$

$$\dot{\alpha} = \Gamma_1^T \omega_g + \Gamma_2^T D_l^{-1}(-P_L - d_l - \Gamma_2 A T A^T \alpha) \tag{9b}$$

$$\dot{P}_M = T_{TG}^{-1}(P_C - P_M - R^{-1}\omega_g) \tag{9c}$$

Control input dynamics:

$$\dot{P}_C = K_{P_C}(R(P_M - P_C) - \gamma(C'(P_C) + \zeta + \mu^+ - \mu^-)) \tag{9d}$$

$$\dot{P}_L = K_{P_L}(D_l^{-1}(-P_L - d_l - \Gamma_2 A T A^T \alpha) + \gamma(U'(P_L) + \lambda - \nu^+ + \nu^-)) \tag{9e}$$

Ancillary variable dynamics:

$$\dot{\theta} = K_\theta A_0 T(A^T \Gamma_1^T \zeta + A^T \Gamma_2^T \lambda - l^+ + l^-) \tag{9f}$$

$$\dot{\zeta} = K_\zeta(P_C - d_g - \Gamma_1 A T A_0^T \theta) \tag{9g}$$

$$\dot{\lambda} = K_\lambda(-P_L - d_l - \Gamma_2 A T A_0^T \theta) \tag{9h}$$

$$\dot{\mu}^+ = K_{\mu^+}(P_C - P_C^{\max})^+_{\mu^+} \tag{9i}$$

$$\dot{\mu}^- = K_{\mu^-}(P_C^{\min} - P_C)^+_{\mu^-} \tag{9j}$$

$$\dot{\nu}^+ = K_{\nu^+}(P_L - P_L^{\max})^+_{\nu^+} \tag{9k}$$

$$\dot{\nu}^- = K_{\nu^-}(P_L^{\min} - P_L)^+_{\nu^-} \tag{9l}$$

$$\dot{l}^+ = K_{l^+}(T A_0^T \theta - P_{TC}^{\max})^+_{l^+} \tag{9m}$$

$$\dot{l}^- = K_{l^-}(P_{TC}^{\min} - T A_0^T \theta)^+_{l^-} \tag{9n}$$

where $K_{P_C}, K_\zeta, K_{\mu^+}, K_{\mu^-}, K_{P_L}, K_\lambda, K_{\nu^+}, K_{\nu^-}, K_\theta, K_{l^+}, K_{l^-}$ are positive diagonal matrices, all representing the controller gains. Also for simplicity, we have used vector forms of positive projection in (9i)-(9n).

In Equations (9g)-(9h), the information of d_g, d_l is needed. Since the disturbance injection is usually uncertain and/or hard to measure, we modify these two equations so that the implementation of the above controller is independent of d_g, d_l:

$$\dot{\zeta} = K_\zeta(M\dot{\omega}_g + D_g\omega_g + P_C - P_M + \Gamma_1 A T A^T \alpha - \Gamma_1 A T A_0^T \theta) \tag{10a}$$

$$\dot{\lambda} = K_\lambda(D_l \omega_l + \Gamma_2 A T A^T \alpha - \Gamma_2 A T A_0^T \theta) \tag{10b}$$

where we have substituted system dynamics (3) into (9g)-(9h). It is important to note that controller (9d)-(9n) is completely distributed, i.e., states are updated using only local information and signals from their neighborhood.

An alternative way is to use Lemma 3 in the control modification, in which ancillary decision vectors $\hat{P}_C, \hat{P}_L, \hat{\theta}, \hat{\zeta}, \hat{\lambda}$ will be introduced. The corresponding extra dynamics can improve the performance and robustness of the closed-loop system, as illustrated in [5], [6].

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