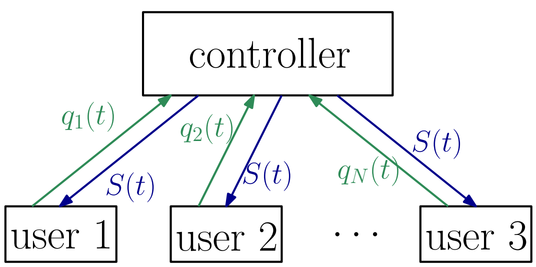
Distributed Proportionally Fair Power Allocation in HVAC Systems with Time-Varying Utilities

Kathryn Heal, Sindri Magnusson, Chinwendu Enyioha

I. Introduction

We study the problem of optimally allocating power resources to a group of users (households), where the relationship between the needs of the power plant and the users may vary with time, and where a measure of fairness in the allocation policy is necessary. Our model focuses on the Heating, Ventilation and Air Conditioning (HVAC) devices of these households. We present a distributed algorithm to solve the resource allocation problem that guarantees proportional fairness in its allocation policy.



Caption

Resource allocation problems in power networks have been studied extensively in the past. Some existing works examine models with stochastic pricing schemes, multiple suppliers, or multiple resources **CITE**. Others propose *batch* algorithms to solve social welfare maximization problems, meaning that the optimal demands for each day are calculated at the beginning of the day and executed over the course of the day [1]. The capabilities of these works are essential components of any comprehensive energy management policy, but in the authors' opinion there remains a frontier that has not yet been adequately addressed.

In real HVAC systems, the demands of the users may vary with time. For instance, the utility function in a heating strategy could be a function of the outdoor temperature; it could also depend on what times of the day the building is occupied.

Most works in the demand response canon require the supplier to have full knowledge of the utility functions of its users. This assumption presents a hurdle in real power distribution networks where communication between power plants and end users is constrained and rather costly. This assumption is particularly prohibitive if (as in our paper) each utility function itself is changing over time. To alleviate this burden, we wonder: if the power plant is allowed to communicate a *few bits* or even just a *single bit* of information to the users, how can limited power resources be fairly and optimally allocated to users with differing energy needs?

Power lines are the primary media through which power plants communicate with their end users; that is, *non*-power information (for instance cost for the next cycle), is sent via the same channel used to supply the power. This method poses a number of challenges: in addition to the fact that they were not designed for communication, security and interference are major concerns **CITE**. Though wireless communication is becoming increasingly reliable, availability of bandwidth is a major constraint. These challenges present the need for new paradigms for distributed optimization and resource allocation strategies that are communication-efficient. In this paper, we assume only a *one-way* communication from the power plant to the users exists and that only a *single bit* of information can be transmitted. Based on this, we present a novel, distributed proportionally fair power allocation policy for temperature control amongst a number of heterogeneous users with differing power needs.

Our system is comprised of a power plant with capacity Q and a set of N heterogeneous users. The heterogeneity of the users lies in the fact that the building sizes, insulation efficiency, heat transfer coefficient and power consumption are allowed to vary. The overarching system objective is to regulate the ambient temperature of the building within some desired range for occupants of the buildings. Let us suppose that the buildings are equipped with (non-identical) heating and cooling devices. A system operator, on a cold day, might wish to guarantee the *maximum* possible common temperature to the buildings (on a hot day, the minimal possible temperature). However, if the capacity Q is very large, this could lead to overheating.

To solve this problem, we let each user specify their desired temperature T_i^{comf}. We consider any excess of this temperature to be wasteful, as it is providing more heat than the user actually wants. Thus our algorithm proposes a strategy that is asymptotically bounded above this desired temperature.

Of course, to achieve a common temperature between the heterogeneous buildings the power plant might need to provide a different amount of power to each user. We apply techniques from the communications and energy management literature [2], [3], [4], [1] to the problem of resource allocation in HVAC systems.

In this paper, we establish an analytic foundation upon which we can eventually incorporate other nonlinear dynamics and further communication restrictions. In the first part of the paper, we consider the stylized case where the power plant is assumed to have knowledge of the needs of the users; that is, it knows the form of each user's utility function. However, we do not require the power plant to dictate to each user how much power to consume. As in [2], we only require that a single 'optimal' value be broadcast uniformly to all users, to reduce communication overhead on the power lines. In the second part of the paper, we propose a demand response algorithm for the varying degrees of restriction on communication. The novelty of our methods is that they can be used with time-varying utility functions.

The rest of the paper is organized as follows: in Section II a model for the indoor temperature of a given building i is introduced. In Section III, we introduce a standard system optimization problem, which uses the *long-term fairness* and the time-varying utility function for each building that are defined in Section IV. The system problem can be solved in a distributed manner once it has been broken into a collection of local problems, as described in Section V. **FINISH**

CHECK: (1) Diagram of the setup (2) What would the perfect problem be? Why can't we do that perfect problem? What assumptions do we make instead?

The objective of this work is to provide the foundation of what will be a series of HVAC systems analysis.

II. Problem Formulation

Consider the HVAC system that regulates the temperature for building $i \in \mathcal{A}$. Let $T_i^{in}(t) = T_i^{in}(q_i(t))$ and T_i^{out} be the temperature inside and outside of the ith household, respectively. We assume that T_i^{out} takes on the same value for all of the users; perhaps the buildings are located relatively near to one another. Furthermore, we assume that we have prior knowledge of the outdoor temperature. We can express the inside temperature [1] as

$$T_i^{in}(t) = T_i^{in}(t-1) + \alpha_i[T_i^{out} - T_i^{in}(t-1)] + \beta_i q_i(t) \tag{1}$$

Make all of the t's $t-1$'s for our discrete case??? Ask Lina. where $\beta_i > 0$ if HVAC appliance i is a heater and $\beta_i < 0$ if appliance i is an air conditioner. In the following sections, we will assume a cold day, where the buildings must be heated; however, our analysis can be easily extended to a hot day.

For ease of notation, we define the shorthand

$$\begin{aligned} T_i^t &= T_i^{in}(t-1) + \alpha_i[T_i^{out} - T_i^{in}(t-1)] \\ &= (1-\alpha_i)T_i^{in}(t-1) + \alpha_i T_i^{out}. \end{aligned} \tag{2}$$

In the definition above, q_i is the load vector for user i; that is, q_i=$(q_i(0), q_i(1), \dots, q_i(t), \dots)$. We emphasize that $T_i^{in}(t)$ is a linear function of $q_i(t)$ for every i and every t. We will assume that the outdoor environment is much "larger" than inside the household, which will mean that the effect of the heat transfer on T_i^{out} is negligible. Thus for every i and every t, T_i^{out} does *not* depend on $T_i^{in}(t)$.

For any of these distributed algorithms to succeed, the controller and the users must agree on a set of initial values and constants, perhaps excluding α_i, and β_i. In this paper, we would regard a controller's "omniscience" as being equivalent to the controller knowing the values of α_i, β_i for all $i \in \mathcal{A}$.

Assumption 1. *Without much loss of generality, we will assume for the remainder of this paper that it is a cold day. By this we generally mean that the temperature is half of what is considered comfortable.*

It follows that for our derivations and simulations, $\beta > 0$. We will now introduce our notion of *long-term fairness* of the system.

Definition 1. *We say that the exponentially-weighted average of the indoor temperature of user i at time t is*

$$\rho_i(t+1) = \frac{1}{m}\left((m-1)\rho_i(t) + T_i^{in}(t)\right) \tag{3}$$

Definition 2. *We say that an energy management strategy is long-term fair if*

$$\lim_{t\to\infty} \rho_i(t) = C \tag{4}$$

where C is the maximal possible temperature that can be guaranteed to all *rooms uniformly.*

Such an energy strategy would be fair in the sense that every room in the group would experience the same indoor temperature, regardless of the physical differences of the buildings.

A. Algorithm 1

We present the resource allocation algorithm for the ideal case. Line 11 is implicit in the ideal algorithm; the controller is assumed to know the full form of the utility functions, and can therefore calculate all of these variables itself.

1: **procedure** HVAC Ideal Algorithm

receive: $T_i^{out}, T_i^{\text{comf}}\ \forall i$
initialize: $T_i^{in}(1), \rho_i(1)\ \forall i$, tol
while $||\mathbf{q} - \mathbf{q}^*||_1$ >tol **do**
  controller computes $S(t)$ as in (13)
  controller broadcasts $S(t)$ to users
  for $i : 1 \rightarrow N$ **do**
    user i updates $\rho_i(t)$ as in (3)
    user i updates $q_i(t)$ as in (9)
    user i updates $T_i^{in}(t)$ as in (1)
    user i "broadcasts" all variables to controller
  $t \leftarrow t + 1$
  goto *loop*.

III. The System Problem

Let $U_i(T_i^{in}(t))$ be a function that is concave and continuously differentiable. We specified in the previous section that our appliance of focus is a heater, so $\beta_i > 0$ for all $i \in \mathcal{A}$. Let $\overline{q} > 0$ be the maximum possible demand for each user, and let $N = |\mathcal{A}|$. For each t, the system problem is

$$\begin{aligned} \max_{q_1(t), q_2(t), \ldots, q_N(t)} \ & \sum_{i=1}^{N} U_i\left(t\right) \\ \text{s.t. } & 0 \leq q_i(t) \leq \overline{q_i} \ \ \forall i \in \mathcal{A} \\ & \sum_{i=1}^{N} q_i(t) \leq Q. \end{aligned} \tag{5}$$

IV. The (distributed) local problem

Let $U_i(t, q(t))$ be a function that is continuously differentiable and concave. For each t, every user $i \in \mathcal{A}$ solves the local problem

$$\begin{aligned} \max_{q_i(t)} \ & U_i\left(t\right) - S(t) q_i(t) \\ \text{s.t. } & 0 \leq q_i(t) \leq \overline{q}. \end{aligned} \tag{6}$$

where $S(t)$ denotes the price per unit power for the community at time t. Notice that T_i^t does not depend on current values of $q_i(t)$ or $T_i^{in}(t)$.

A. Utility function

Our utility function is

$$U_i(t, q(t)) := 2 g_i ln \left(1 + \frac{T_i^{in}(q_i(t))}{\rho_i(t)}\right) \tag{7}$$

where g_i is a guarantee parameter assigned to each user i. In future applications this guarantee parameter may be assigned differently, but in this context we will generally assume that $g_i = 1/\beta_i$ (see Assumption 2). This form of utility function is desirable [5] because it is strongly concave and exhibits valuable long-term fairness properties [2], which we will define next. Furthermore, it features parameter m that we can tailor to enhance the performance of our algorithm.

B. The ideal case

We show that we can find an explicit expression for each optimal demand response $q_i(t)$ in the ideal case. The following theorem and its proof are adapted for our purposes from Lemma 1 of [2].

Lemma 1. *If the set of feasible rates for each user i is $\Gamma_i = [0, \overline{q}]$, then the minimum $S(t) \geq 0$ that satisfies*

$$\begin{aligned} \sum_{i=1}^{N} \frac{2}{S(t)} - \frac{1}{\beta_i} \big(\rho_i(t) + (1 - \alpha_i) T_i^{in}(t-1) \\ + \alpha_i T_i^{out}\big) \leq Q \end{aligned} \tag{8}$$

causes the solution to the local optimization problem (6) *to be the same as that to the system optimization problem* (5)*. Then the optimal demands for each user are*

$$q_i^*(t) = \left[\frac{2g_i}{S(t)} - \frac{1}{\beta_i}\left(\rho_i(t) + T_i^t\right)\right]_0^{\overline{q}}, \tag{9}$$

where we use the notation $[\,\cdot\,]_a^b := \min\{b, \max\{a, \ \cdot\,\}\}$*, and T_i^t is as defined in Equation* (1)*.*

Proof. Let bold-face characters denote the collection of primal or dual variables for the N users; for example, $\mathbf{q} := (q_1(t), \ldots, q_N(t))$. The Lagrangian $L(\mathbf{q}, \lambda, \mu, \sigma)$ of the system problem is

$$\begin{aligned} & \sum_{i=1}^{N} 2 g_i ln \left(1 + \frac{T_i^{in}(q_i(t))}{\rho_i(t)}\right) - \lambda \left(\sum_{i=1}^{N} q_i(t) - Q\right) \\ & \quad + \sum_{i=1}^{N} \mu_i q_i(t) - \sum_{i=1}^{N} \sigma_i (q_i(t) - \overline{q}), \end{aligned} \tag{10}$$

where the dual variables $\lambda \geq 0, \mu_i \geq 0$ and $\sigma_i \geq 0$ for all i. Taking the derivative of (10) with respect to $q_i(t)$ and setting it to zero,

$$\begin{aligned} \frac{dL}{dq_i(t)} &= \frac{dL}{dT_i^{in}(t)} \frac{dT_i^{in}}{dq_i(t)} + \frac{dL}{dq_i(t)} = 0 \\ \frac{dL}{dq_i(t)} &= \frac{2 g_i \beta_i}{T_i^{in}(q_i(t)) + \rho_i(t)} - \lambda + \mu_i - \sigma_i = 0. \end{aligned} \tag{11}$$

Rearranging this, we get

$$\beta_i q_i(t) = \frac{2 g_i \beta_i}{\lambda - \mu_i + \sigma_i} - \rho_i(t) - T_i^t. \tag{12}$$

By the KKT conditions, $\mu_i > 0$ only if $q_i(t) = 0$, and $\sigma_i > 0$ only if $q_i(t) = \overline{q}$. By setting $S(t) = \lambda$, the KKT conditions imply that if $\frac{2g_i}{S(t)} - \frac{1}{\beta_i}\left(\rho_i(t) + T_i^t\right) < 0$, then $\mu_i > 0$ and $q_i(t) = 0$. Also, if $\frac{2g_i}{S(t)} - \frac{1}{\beta_i}\left(\rho_i(t) + T_i^t\right) > 0$, then $\sigma_i > 0$ and $q_i(t) = \overline{q}$. This implies that the optimal demand will be (9), and that $S(t) \geq 0$ is the minimum value that satisfies (8). ☐

In the case where the feasible demands are unbounded above, i.e. $\Gamma_i = \mathbb{R}_+$, we can give an explicit formula for the optimal coordinating variable $S(t)$. This will allow us to develop some intuition for the dynamics of the system, from which point we may impose additional assumptions.

Lemma 2. *If each $\overline{q}_i = \infty$, this optimal $S(t)$ is*

$$S(t) = \frac{2\sum_{j=1}^N g_j}{Q + \sum_{i=1}^N \frac{1}{\beta_i}\left(\rho_i(t) + T_i^t\right)}. \tag{13}$$

Proof. This proof is adapted from that of Lemma 2 of [2] to accommodate HVAC systems. Following the steps in that paper, one can show that there exists a time t_0 such that, for all $t \geq t_0$, we have $\frac{2g_i}{S(t)} - \frac{1}{\beta_i}\left(\rho_i(t) + T_i^t\right) \geq 0$. For the sake of brevity, we will omit those calculations in this work.

Since $\overline{q} = \infty$, both the upper and lower limits in (9) are avoided for all $t \geq t_0$. Therefore, Eq. (8) is tight and

$$\begin{aligned}\sum_{i=1}^N \frac{2g_i}{S(t)} - \frac{1}{\beta_i}\big(\rho_i(t) + (1-\alpha_i)T_i^{in}(t-1) \\ +\alpha_i T_i^{out}\big) = Q \\ \frac{2\sum_{j=1}^N g_j}{S(t)} = Q + \sum_{i=1}^N \frac{1}{\beta_i}\big(\rho_i(t) \\ +(1-\alpha_i)T_i^{in}(t-1) + \alpha_i T_i^{out}\big).\end{aligned} \tag{14}$$

That is, when $S(t)$ is defined as in (13), the solution $\{q_1^*(t), \dots, q_1^*(t)\}$ to the local optimization problem is identical to that of the system optimization problem (i.e. is the optimal dual variable for the limited feedback cases). ☐

Again, we can only get this explicit, closed-form formula for $S(t)$ because our feasible range is unbounded. The bounded case is addressed in Section V-D.

V. Convergence in the Ideal Case

Consider $\Gamma_i = \mathbb{R}^+$. For ease of computation and stability analysis, we express the system above in its recursive form. This allows us to examine the immediate effect that the temperature and its average exponential weight have on one another, and to rewrite this as a linear system.

$$\begin{aligned}\rho_i(t+1) &= \left(\frac{m-1}{m}\right)\rho_i(t) + \left(\frac{1}{m}\right)T_i^{in}(t) \\ T_i^{in}(t+1) &= \rho_i(t)\left(-\frac{m-1}{m}\right) \\ &+ \frac{\beta_i(m-1)}{m}\frac{g_i}{\sum_{j=1}^N g_j}\sum_{j=1}^N\left(\frac{1}{\beta_j}\right)\rho_j(t) \\ &+ T_i^{in}(t)\left(-\frac{1}{m}\right) + \beta_i\frac{g_i}{\sum_{j=1}^N g_j}\sum_{j=1}^N\Big(\frac{1}{\beta_j m} \\ &+\frac{1-\alpha_j}{\beta_j}\Big)T_j^{in}(t) \\ &+ \frac{\beta_i g_i}{\sum_{j=1}^N g_j}\left(Q + \sum_{j=1}^N \frac{\alpha_j}{\beta_j}T_j^{out}\right).\end{aligned} \tag{15}$$

Assumption 2. *At time $t = 0$, each user i and the supplier agree on the value of β_i, and the supplier sets each $g_i = 1/\beta_i$.*

A. Presenting the linear system

Let $\mathbf{x}(t) = (\rho_1(t), \dots, \rho_N(t), T_1^{in}(t), \dots, T_N^{in}(t))^T$. Define the vectors $\mathbf{a} = (\alpha_1, \dots, \alpha_N)^T$, $\mathbf{b} = (\beta_1, \dots, \beta_N)^T$, $\mathbf{e} = (1, \dots, 1)^T \in \mathbb{R}^N$, $\mathbf{T} = (T_1^{out}, \dots, T_N^{out})^T$, and $\mathbf{g} = (g_1, \dots, g_N)^T$, where the g_i are rate guarantees. Define the matrices $\mathbf{A} = diag(\mathbf{a})$, $\mathbf{B} = diag(\mathbf{b})$, and $\mathbf{G} = diag(\mathbf{g})$. Then if $\mathbf{G} = \mathbf{B}^{-1}$ as in Assumption 2,

$$\mathbf{C} = \left(\frac{m-1}{m}\right)\left(-\mathbf{I}_N + \frac{1}{\mathbf{e}^T\mathbf{g}}\mathbf{G}\mathbf{b}\mathbf{e}^T\mathbf{B}^{-1}\right)$$

$$\mathbf{D} = \left(-\frac{1}{m}\right)\mathbf{I}_N + \frac{1}{\mathbf{e}^T\mathbf{g}}\mathbf{G}\mathbf{b}\mathbf{e}^T\left(\left(\frac{1}{m}+1\right)\mathbf{B}^{-1} - \mathbf{A}\mathbf{B}^{-1}\right)$$

$$\mathbf{F} = \frac{1}{\mathbf{e}^T\mathbf{g}}\mathbf{G}\mathbf{b}\left(Q + \mathbf{T}^T\mathbf{B}^{-1}\mathbf{a}\right)$$

and we will have

$$\mathbf{x}(t+1) = \underbrace{\begin{bmatrix} \frac{m-1}{m}\mathbf{I}_N & \frac{1}{m}\mathbf{I}_N \\ \mathbf{C} & \mathbf{D}\end{bmatrix}}_{:=\Lambda}\mathbf{x}(t) + \begin{bmatrix}\mathbf{0}_{N\times 1} \\ \mathbf{F}\end{bmatrix}.$$

Notice that $\Lambda \in \mathbb{R}_{2N\times 2N}$ is a block matrix consisting of the aforementioned $N \times N$ matrices.

B. Convergence of the linear system

For ease of notation, let

$$\zeta_k := \frac{\sum_{j=1}^N \beta_j}{\beta_k}\frac{g_k}{\sum_{j=1}^N g_j} \tag{16}$$

where $k = 1, \dots, N$. Let P_A be chosen as in Lemma 5, and P_B be chosen as in Lemma 6.

Proposition 1. *If $\mathbf{g}, \mathbf{b}$ are such that the vector $\zeta \neq \mathbf{e}$, and m is chosen so that*

$$1 < \min \left\{ \left| \frac{1}{(\zeta_{P_A} - 1)(m-1)} \right|, \frac{1 - \left| \frac{1}{m} + \zeta_{P_B}(1 - \alpha_{P_B}) \right|}{\frac{1}{m} |\zeta_{P_B} - 1|} \right\}, \tag{17}$$

then the system defined by Λ is stable.

Proof. If Lemmas (5) and (6) in the Appendix both hold, then

$$\begin{aligned} &|eig(\Lambda)| \leq ||\Lambda||_1 \\ &= \max \left\{ \left| \frac{m-1}{m} - \frac{m-1}{m} + \frac{m-1}{m}\zeta_{P_A} \right|, \right. \\ &\qquad \left. \left| \frac{1}{m} - \frac{1}{m} + \zeta_{P_B}\left(\frac{1}{m} + 1 - \alpha_{P_B}\right) \right| \right\} \\ &< \max\{1, 1\} = 1. \end{aligned} \tag{18}$$

This implies that the system is stable. □

Corollary 1. *If $\mathbf{g}, \mathbf{b}$ are such that the vector $\zeta = \mathbf{e}$, and m is chosen so that*

$$m > \max_k \left\{ \frac{1}{1 - |1 - \alpha_k|} \right\} \tag{19}$$

then the system defined by Λ is stable.

The proof of Corollary 1 follows in the same format as that of Proposition 1. If $\zeta = \mathbf{e}$, then every element in ζ_{P_A} and ζ_{P_B} will be equal to 1. If we choose m as described above, then $|1 + 1/m - \alpha_{P_B}| < 1$, and

$$\begin{aligned} &|eig(\Lambda)| \leq ||\Lambda||_1 \\ &= \max \left\{ \left| \frac{m-1}{m} \right|, \left| \frac{1}{m} + 1 - \alpha_{P_B} \right| \right\} \\ &< \max\{1, 1\} = 1 \end{aligned}$$

under the condition that $|1+1/m-\alpha_{P_B}| < 1$. This can be achieved by selecting m that is far larger than α_{P_B}.

C. Fixed points.

All values in this linear system should share a common fixed point because of the "maximum heat subject to Q" criteria that we have used to define long-term fairness in Equation (4). Indeed, we can write this fixed point explicitly as $(T^{in})^*$, below. The proof of the following proposition can be found in the appendix.

Proposition 2. *Let each $g_i := 1/\beta_i$ as in Assumption 2, and let $\Gamma = \mathbb{R}^+$. If we define*

$$(T^{in})^* = \frac{Q + \sum_{j=1}^{N} \frac{\alpha_j}{\beta_j} T_j^{out}}{\sum_{j=1}^{N} \frac{\alpha_j}{\beta_j}}, \tag{20}$$

then $(T^{in})^*$ is a fixed point for each of the functions $\rho_1, \ldots, \rho_N, T_1^{in}, \ldots,$ and T_N^{in}.

This result, along with the stability of the system, implies that

$$\lim_{t\to\infty} \rho_i(t) = \lim_{t\to\infty} T_i^{in}(t) = (T^{in})^*. \tag{21}$$

D. Restricting the set Γ_i of feasible demands

We now suppose that the maximum feasible demands $\overline{q}_1, \ldots, \overline{q}_N$ may be finite and unique to each user.

Lemma 3. *Let $\Gamma_i = [0, \overline{q}_i]$ for each i. Let $D_1, \ldots, D_N$ take arbitrary values that are constant with respect to t, for which $\sum_{i=1}^N \alpha_i D_i > 0$. If $\sum_{i=1}^N \overline{q}_i > Q$, then using Algorithm 1 we have*

$$\lim_{t\to\infty} \sum_{i=1}^{N} D_i \rho_i(t) \geq \frac{\sum_{i=1}^N D_i \sum_{i=1}^N (\alpha_i T_i^{out})}{\sum_{i=1}^N (\alpha_i D_i)}. \tag{22}$$

Proof. We begin with

$$\lim_{t\to\infty} \sum_{i=1}^{N} D_i \rho_i(t) = \sum_{i=1}^{N} D_i \lim_{t\to\infty} \rho_i(t) \tag{23}$$

Recall our assumption here that $\overline{q}$ is finite. Thus Equation (20) need not be a fixed point, and we must find another limit value for this restricted case. By Equation (21),

$$\begin{aligned} &\sum_{i=1}^{N} D_i \lim_{t\to\infty} \rho_i(t) = \sum_{i=1}^{N} D_i \lim_{t\to\infty} T_i^{in}(t) \\ &\geq \sum_{i=1}^{N} D_i \lim_{t\to\infty} \left((1-\alpha_i) T_i^{in}(t-1) + \underbrace{\alpha_i T_i^{out} + 0}_{F_i} \right) \\ &= \sum_{i=1}^{N} \left(D_i \lim_{t\to\infty} (1-\alpha_i) T_i^{in}(t-1) + F_i \right) \\ &= \sum_{i=1}^{N} D_i (1-\alpha_i) \lim_{t\to\infty} T_i^{in}(t-1) + \sum_{i=1}^{N} F_i \end{aligned} \tag{24}$$

The inequality follows from our assumption that $\beta_i > 0$. Notice that F_i is constant with respect to t, so is unaffected by the limit. Then if $T^* = \lim_{t\to\infty} T_i^{in}(t)$, we have

$$\sum_{i=1}^{N} D_i T^* \geq \sum_{i=1}^{N} D_i (1-\alpha_i) T^* + \sum_{i=1}^{N} F_i \tag{25}$$

so

$$T^* \left(\sum_{i=1}^{N} \alpha_i D_i \right) \geq \sum_{i=1}^{N} F_i. \tag{26}$$

By assumption, all $\alpha_i > 0$ and $\sum_{i=1}^N \alpha_i D_i > 0$, so

$$T^* \geq \frac{\sum_{i=1}^N F_i}{\left(\sum_{i=1}^N \alpha_i D_i\right)}. \tag{27}$$

Therefore

$$\lim_{t\to\infty} \sum_{i=1}^N D_i \rho_i(t) = \sum_{i=1}^N D_i T^* \geq \frac{\sum_{i=1}^N D_i \sum_{i=1}^N F_i}{\sum_{i=1}^N (\alpha_i D_i)}, \tag{28}$$

as desired. □

Lemma 4. *For α_i scalar and N finite,*

$$\lim_{t\to\infty} \sum_{i=1}^N T_i^t = \sum_{i=1}^N (1-\alpha_i) \lim_{t\to\infty} \rho_i(t) + \sum_{i=1}^N \alpha_i T_i^{out} \tag{29}$$

$$\lim_{t\to\infty} \sum_{i=1}^N T_i^t = \lim_{t\to\infty} \sum_{i=1}^N (1-\alpha_i) T_i^{in}(t-1) + \alpha_i T_i^{out} \tag{30}$$

Since N is finite, we can move the limit inside the sum.

$$\begin{aligned} \lim_{t\to\infty} \sum_{i=1}^N T_i^t &= \sum_{i=1}^N \left((1-\alpha_i) \lim_{t\to\infty} T_i^{in}(t-1) + \alpha_i T_i^{out}\right) \\ &= \sum_{i=1}^N (1-\alpha_i) \lim_{t\to\infty} \rho_i(t) + \sum_{i=1}^N \alpha_i T_i^{out} \end{aligned} \tag{31}$$

VI. SHOWING CONVERGENCE TO A FIXED POINT OF ρ WHEN $\overline{q}$ IS FINITE.

The next theorem imposes a condition on all $\overline{q}_i$ that ensures that each room's temperature never *exceeds* some pre-defined comfortable temperature. This way, the rooms do not become uncomfortably hot. This crucial feature outweighs the cost of possibly not using all of the available resources Q.

Theorem 1. *Let T_i^{comf} be a "comfortable" temperature as defined by user i. Define $q_i^{comf} := \alpha_i(T_i^{comf} - T_i^{out})/\beta_i$. Suppose that each*

$$\Gamma_i := \left[0, \min\left\{\overline{q}_i, q_i^{comf}\right\}\right], \tag{32}$$

where $\overline{q}_i < \infty$. Let $\mathcal{I}$ denote the set of i such that $\overline{q}_i \geq q_i^{comf}$, and let $(T^{in})^*_{\mathcal{I}}$ be the revised fixed point

$$(T^{in})^*_{\mathcal{I}} := \frac{Q - \sum_{i\notin\mathcal{I}} \overline{q}_i + \sum_{i\in\mathcal{I}} \frac{\alpha_i T_i^{out}}{\beta_i}}{\sum_{i\in\mathcal{I}} \frac{\alpha_i}{\beta_i}}. \tag{33}$$

Then Algorithm 1 will provide a strategy with the following asymptotic guarantees. For all $i \in \mathcal{I}$,

$$\lim_{t\to\infty} T_i^{in}(t) = \min\left\{T_i^{comf}, (T^{in})^*_{\mathcal{I}}\right\}. \tag{34}$$

For all $i \notin \mathcal{I}$ (that is, i such that $\overline{q}_i < q_i^{comf}$),

$$\lim_{t\to\infty} T_i^{in}(t) = T_i^{out} + \frac{\beta_i \overline{q}_i}{\alpha_i}. \tag{35}$$

Proof. To prove this theorem we can assume that there exists some $i \neq j$ for which $\overline{q}_i \geq q_i^{\text{comf}}$ and $\overline{q}_j \geq q_j^{\text{comf}}$; if not, then this problem would be trivial. For simplicity, suppose that the users are ordered such that for some $h^* \geq 2$, we have $\mathcal{I} = \{1, \dots, h^*-1\}$. Then its complement is $\mathcal{I}^c = \{h^*, \dots, N\}$.

We prove this by induction on i. We first consider the case when $i = 1$. With some minor adjustments to the proof in [2], we can show that there exists a t_0 for which

$$\frac{2}{\beta_i S(t)} - \frac{1}{\beta_i}\left(\rho_i(t) + T_i^t\right) \geq 0. \tag{36}$$

As $t \to \infty$, Algorithm 1 will select $S(t)$ so that

$$\begin{aligned} Q &= \sum_{i=1}^N \min\left\{\frac{2}{\beta_i S(t)} - \frac{1}{\beta_i}\left(\rho_i(t) + T_i^t\right), \overline{q}_i\right\} \\ &\leq \sum_{i=1}^N \frac{1}{\beta_i}\left(\frac{2}{S(t)} - \left(\rho_i(t) + T_i^t\right)\right) \\ &\leq \frac{1}{S(t)} \sum_{i=1}^N \frac{2}{\beta_i} - \sum_{i=1}^N \frac{\rho_i(t) + T_i^t}{\beta_i} \end{aligned} \tag{37}$$

Lemma 4 tells us that as $t \to \infty$,

$$\begin{aligned} Q &\leq \frac{1}{S(t)} \sum_{i=1}^N \frac{2}{\beta_i} - \sum_{i=1}^N \frac{\rho_i(t) - (1-\alpha_i)\rho_i + \alpha_i T_i^{out}}{\beta_i} \\ &= \frac{1}{S(t)} \sum_{i=1}^N \frac{2}{\beta_i} - \sum_{i=1}^N \frac{\alpha_i}{\beta_i}(\rho_i + T_i^{out}) \end{aligned} \tag{38}$$

Thus

$$Q + \sum_{i=1}^N \frac{\alpha_i}{\beta_i}(\rho_i + T_i^{out}) \leq \frac{1}{S(t)} \sum_{i=1}^N \frac{2}{\beta_i} \tag{39}$$

and since we assume positive β_j, this leaves

$$\frac{1}{S(t)} \geq \frac{1}{\sum_{i=1}^N \frac{2}{\beta_i}} \left(Q + \sum_{i=1}^N \frac{\alpha_i}{\beta_i}(\rho_i + T_i^{out})\right). \tag{40}$$

For convenience, let us rewrite Equation (42) using temporary notation. We will not continue this notation outside of this proof. Here, (42) becomes

$$\frac{1}{S(t)} \geq A\left(Q + \sum_{i=1}^N (B_i \rho_i + C_i)\right). \tag{41}$$

By Lemma 3, as $t \to \infty$ the right hand side is bounded above by

$$\frac{1}{S(t)} \geq A\left(Q + \frac{\sum_{i=1}^N B_i \sum_{i=1}^N (\alpha_i T_i^{out})}{\sum_{i=1}^N (\alpha_i B_i)} + \sum_{i=1}^N C_i\right). \tag{42}$$

...

Must finish this.

Sketch of the rest: The first limit is a direct result of the preceding lemma. The second limit follows from Equation (1). In the case where $i \notin \mathcal{I}$, we would set $T = \lim_{t\to\infty} T_i^{in}(t)$ and $q_i(t) \equiv \overline{q}_i$ since the little machine is clearly struggling to take as much power as it possibly can!

$$T = T + \alpha_i(T_i^{out} - T) + \beta_i \overline{q}_i$$
$$\alpha_i T = \alpha_i T_i^{out} + \beta_i \overline{q}_i$$

(And of course, $\lim \rho = \lim T_i^{in}$ as before...)

Remark 1. *Proposition 2 is a special case of Theorem 1. In the unbounded case, $\mathcal{I}^c = \emptyset$, so $\sum_{i\notin\mathcal{I}} \overline{q}_i = 0$, and Equation* (20) *is equivalent to Equation* (33).

Theorem 1 states that if a user has the *capability* (e.g. they have a new/strong enough appliance) to reach their comfortable or guaranteed temperature, they will, but if they do not have this capability then they will achieve the maximum temperature that they possibly can. Enforcing $q_i(t) \in [0, \min\{\overline{q}_i, q_i^{\text{comf}}\}]$ keeps the equation feasible, since it asserts an upper bound that does not exceed $\overline{q}_i$, and ensures that each building does not become uncomfortably warm according to some standard set by its occupants.

Thus the scenario is fair in the sense that each user will get the maximal collective temperature, *within reason*. Each user should demand an amount of power that will be comfortable, but we wish to enforce a safety bound so that the occupants are not roasted.

VII. Numerical Results

A. Unbounded Ideal Case

We demonstrate the performance of Algorithm 1 when each $\Gamma_i = \mathbb{R}_+$. For now, we do not incorporate any user-defined notions of comfortable temperatures. This simulation uses the following parameters:

$\gamma = 1$, $m = 11$, $Q = 100$, $T = 100$, $N = 6$, $\alpha = (0.1, \dots, 0.4)^T$, $\beta = (0.5, \dots, 0.3)^T$, $\rho(0) = (40, \dots, 40)^T$, $T^{in}(0) = (50, \dots, 70)^T$, $T^{out} = (40, \dots, 40)$.

Figures 2 and 3 show that in the ideal case the algorithm converges to the fixed point 64.1440, as defined by (20). This shows that even in the worst-case scenario where the capacity Q was initialized to be too small, thus starving the users, each user will suffer fairly. The larger the supply Q, the warmer the rooms can get.

In this particular case, the bound given by (19) would correspond to $m > 10$; indeed, the algorithm converges under this condition. Notice that the feasible sets Γ_i for these simulations are unbounded.

B. Bounded Ideal Case

This simulation uses the following parameters:

$\gamma = 1$, $m = 11$, $Q = 100$, $T = 100$, $N = 6$, $\alpha = (0.1, \dots, 0.4)^T$, $\beta = (0.5, \dots, 0.3)^T$, $\rho(0) = (40, \dots, 40)^T$, $T^{in}(0) = (50, \dots, 70)^T$, $T^{out} = (40, \dots, 40)$, and $\overline{q} = (20, \dots, 20)^T$. Figure 6 shows the case where $T^{\text{comf}} = (70, \dots, 80)^T$. Figure 7 shows the case where $T^{\text{comf}} = (75, \dots, 75)^T$.

VIII. Conclusion

The ideal case is the first step towards integrating a cooperative game strategy into an extensive, flexible framework where each entity (e.g. controller, user) has even less information about the other's needs.

Include Nesterov's Nonlinear Programming to Bibliography.

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IX. Appendix

Recall our definition of ζ_k from Equation (16).

Lemma 5. *Let $P := \arg\max_i \left\{ \left| \zeta_i \right| \right\}$. If m is chosen so that*

$$1 < \left| \frac{1}{(\zeta_P - 1)(m-1)} \right|,$$

then

$$\left| \frac{m-1}{m} \zeta_P \right| < 1.$$

Proof. If m is chosen in this way,

$$\begin{aligned} \left| \zeta_P - 1 \right| &< \frac{1}{m-1} \\ 1 + \left| \zeta_P - 1 \right| &< \frac{m}{m-1} \\ \left| 1 + (\zeta_P - 1) \right| &< \frac{m}{m-1} \\ \left| \zeta_P \right| &< \frac{m}{m-1} \\ \left| m - \zeta_P \right| &< m \\ \left| \frac{m-1}{m} \zeta_P \right| &< 1. \end{aligned}$$

□

Lemma 6. *Now let P be the maximizer of*

$$\max_i \left\{ \left| \zeta_i \left(\frac{1}{m} + 1 - \alpha_i \right) \right| \right\}. \tag{43}$$

If m is chosen so that

$$1 < \frac{1 - \left| \frac{1}{m} + \zeta_P (1 - \alpha_P) \right|}{\frac{1}{m} \left| \zeta_P - 1 \right|}, \tag{44}$$

then

$$\left| \zeta_P \left(\frac{1}{m} + 1 - \alpha_P \right) \right| < 1. \tag{45}$$

Proof. If m is chosen as in Equation (44),

$$\left| \frac{\zeta_P}{m} - \frac{1}{m} \right| < 1 - \left| \frac{1}{m} + \zeta_P (1 - \alpha_P) \right|. \tag{46}$$

Using the triangle inequality, this leaves

$$\left| \zeta_P (1 - \alpha_P) + \frac{\zeta_P}{m} \right| < 1. \tag{47}$$

□

A. *Proof of Proposition 2*

Proof. From (15) we have

$$\begin{aligned} \rho_i^* &= \left(\frac{m-1}{m} \right) \rho_i^* + \left(\frac{1}{m} \right) (T_i^{in})^* \\ &\implies \rho_i^* = (T_i^{in})^* \\ (T_i^{in})^* &= \rho_i^* \left(-\frac{m-1}{m} \right) \\ &\quad + \frac{\beta_i (m-1)}{m} \frac{g_i}{\sum_{j=1}^N g_j} \sum_{j=1}^N \left(\frac{1}{\beta_j} \right) \rho_j^* \\ &\quad + (T_i^{in})^* \left(-\frac{1}{m} \right) + \beta_i \frac{g_i}{\sum_{j=1}^N g_j} \sum_{j=1}^N \left(\frac{1}{\beta_j m} \right. \\ &\quad \left. + \frac{1 - \alpha_j}{\beta_j} \right) (T_j^{in})^* \\ &\quad + \beta_i \frac{g_i}{\sum_{j=1}^N g_j} \left(Q + \sum_{j=1}^N \frac{\alpha_j}{\beta_j} T_j^{out} \right). \end{aligned} \tag{48}$$

Rearranging this, we get

$$\begin{aligned} 2(T_i^{in})^* &= \\ &\sum_{j=1}^N (T_j^{in})^* \left(\frac{\beta_i}{\beta_j} \right) \left(\frac{g_i}{\sum_{j=1}^N g_j} \right) (2 - \alpha_j) \\ &+ \beta_i \frac{g_i}{\sum_{j=1}^N g_j} \left(Q + \sum_{j=1}^N \frac{\alpha_j}{\beta_j} T_j^{out} \right). \end{aligned} \tag{49}$$

There are N equations and N unknowns here, so we can solve for $(T_i^{in})^*$. However, we also know that $(T_i^{in})^* = (T_j^{in})^*$ for all $i, j = 1, \ldots, N$, which greatly simplifies the problem. For ease of notation let $(T^{in})^* := (T_i^{in})^*$.

$$\begin{aligned} (T^{in})^* &\left[2 - \sum_{j=1}^N \left(\frac{\beta_i}{\beta_j} \right) \left(\frac{g_i}{\sum_{j=1}^N g_j} \right) (2 - \alpha_j) \right] \\ &= \beta_i \frac{g_i}{\sum_{j=1}^N g_j} \left(Q + \sum_{j=1}^N \frac{\alpha_j}{\beta_j} T_j^{out} \right) \end{aligned} \tag{50}$$

Assumption 2 gives that $\beta_i g_i = 1$, so we arrive at the final form of $(T_i^{in})^*$ and $(\rho_i)^*$, for all $i \in \mathcal{A}$. □

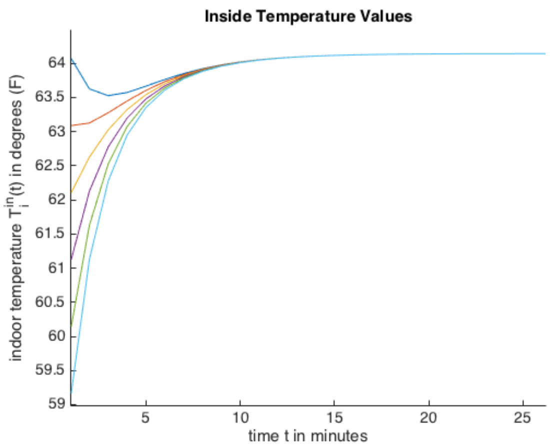

Fig. 1. Values of $T_i^{in}(t)$ in the ideal, unbounded case ($\overline{q} = \infty$).

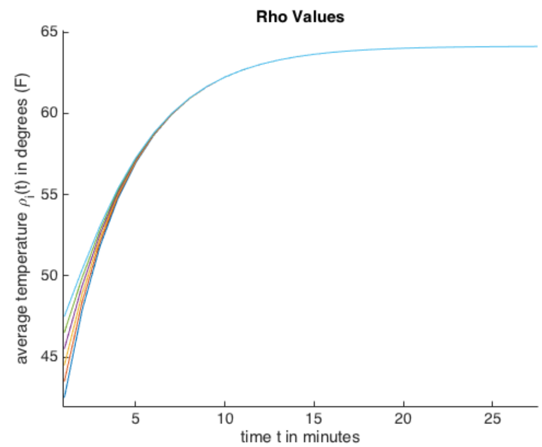

Fig. 2. Values of $\rho_i(t)$ in the ideal, unbounded case ($\overline{q} = \infty$).

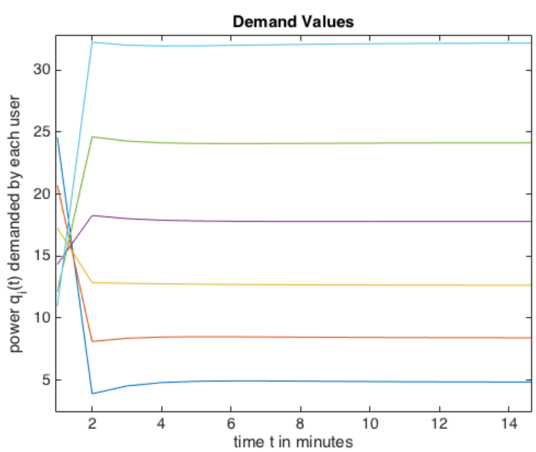

Fig. 3. Values of $q_i(t)$ in the ideal, unbounded case ($\overline{q} = \infty$).

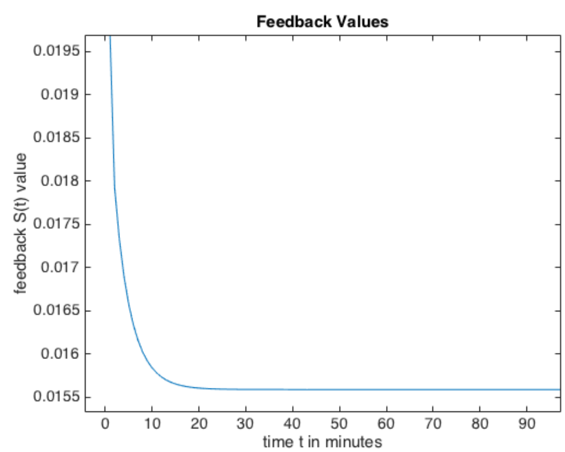

Fig. 4. Values of $S(t)$ in the ideal, unbounded case ($\overline{q} = \infty$).

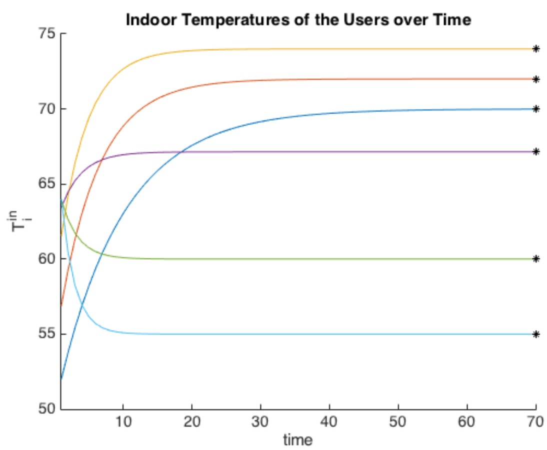

Fig. 5. Values of $T_i^{in}(t)$ in the ideal, bounded case ($\overline{q} < \infty$). The black asterisks indicate the limits predicted by Theorem 1.

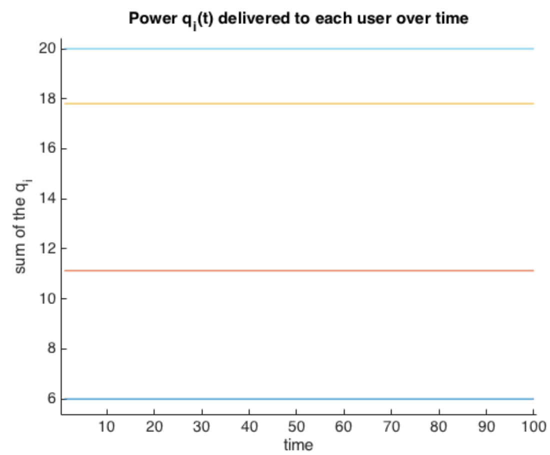

Fig. 6. Values of $\rho_i(t)$ in the ideal, bounded case ($\overline{q} < \infty$). Fix the y-axis labels

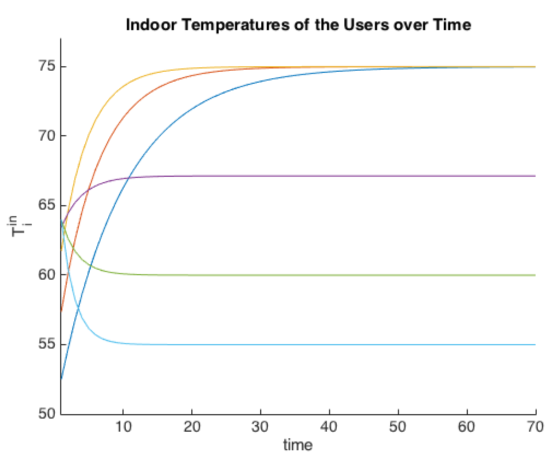

Fig. 7. Values of $T_i^{in}(t)$ in the ideal, bounded case ($\overline{q} < \infty$).

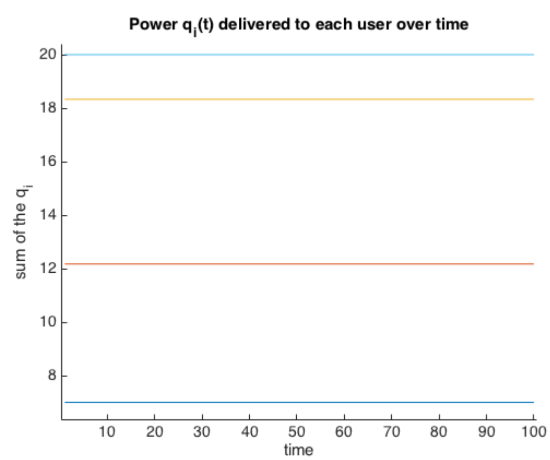

Fig. 8. Values of $S(t)$ in the ideal, bounded case ($\overline{q} < \infty$). Fix the y-axis labels