Exact Convex Relaxation of OPF for Radial Networks using Branch Flow Model

Na Li, Lijun Chen, and Steven H. Low

Abstract—The optimal power flow (OPF) problem is generally nonconvex. Recently a second-order cone relaxation for OPF has been proposed using the branch flow model. In this paper, we provide sufficient conditions under which the relaxation is exact, and demonstrate that these conditions hold for a wide class of practical power distribution systems.

I. INTRODUCTION

In previous work [1], [2], we advocate the use of branch flow models for the design and operation of power systems, including optimal power flow, demand response, and Volt/VAR control. In contrast to bus injection models which focus on nodal variables such as bus current and power injections, branch flow models focus on currents and power flows on individual branches [3], [4]. They have been used mainly for modeling distribution circuits which tend to be radial, but has received far less attention. The optimal power flow (OPF) problem seeks to minimize a certain cost function, such as power loss and generation cost, subject to physical constraints including Kirchoff's laws, thermal constraints, as well as voltage regulation constraints. There has been a great deal of research on OPF since Carpentier's first formulation in 1962 [5]. OPF is generally nonconvex and NP hard, and a large number of optimization algorithms and relaxations have been proposed; see, e.g., [6], [7], [8], [9], [10]. Recently, a semidefinite relaxation (SDR) of OPF is proposed in [11] and a sufficient condition is derived in [12] under which the SDR is exact. This condition is shown to essentially hold in various IEEE test systems. While this line of research has generated a lot of interest, limitations of the SDR have also been studied in [13] using 3, 5, and 7-bus system. Moreover, if SDR fails to provide exact relaxations, the solutions produced by the SDR are physically meaningless in those cases. Remarkably, it turns out that if the network is radial, then the sufficient condition of [12] always holds, provided that the bounds on the power flows satisfy a simple pattern [14], [15], [16]. This is important as almost all distribution systems are radial networks.

Indeed, for radial networks, different convex relaxations have also been studied using branch flow models. The model considered in this paper is first proposed in [3], [4] for the optimal placement and sizing of switched capacitors in distribution circuits for Volt/VAR control. Recasting the model as a set of linear constraints together with a set of quadratic equality

N. Li and S. H. Low are with the Division of Engineering and Applied Science, California Institute of Technology, Pasadena, CA 91125, USA (Emails: {nali, slow}@caltech.edu).

L. Chen is with the College of Engineering and Applied Sciences, University of Colorado, Boulder, CO 80309, USA (Email: lijun.chen@colorado.edu).

constraints, references [17] [1] propose a second-order-cone (SOC) convex relaxation, and prove that the relaxation is exact for radial networks, when there are no upper bounds on the loads. See also [18] for an SOC relaxation of a linear approximation of the branch flow model in [3], [4], and [19], [20], [21] for other branch flow models.

Ignoring upper bounds on the load may be unrealistic, e.g., in the context of demand response. In a previous paper [2], we prove that the SOC relaxation is exact for radial networks, provided there are no upper bounds on the voltage magnitudes and some other sufficient conditions hold. Those sufficient conditions however place strong requirements on the impedance of the distribution lines and on the load and generation patterns in the radial network. In this paper, we propose less restrictive sufficient conditions under which the SOC relaxation is exact. As examples, we show that these conditions hold in two distribution circuits of the Southern California Edison (SCE), with high penetration of photovoltaic (PV) generation. Roughly speaking, these sufficient conditions hold in many real distribution systems where $v \sim 1$ p.u., $p, q < 1$ p.u. , $r, x << 1$ p.u., and $\frac{r}{x}$ is bounded. Here v, p, q are the bus voltage, real power consumption, and reactive power consumption; and r, x are the resistance and reactance of the distribution lines. Moreover, we provide upper bounds on the voltage magnitudes for the SOC relaxation solutions. This would facilitate the voltage regulation in distribution systems.

The paper is organized as follows. We first present the branch flow model in section II. We then provide in section III sufficient conditions under which the SOC relaxation is exact for radial networks when there are no upper bounds on bus voltage magnitudes. Finally, in section IV, we illustrate these sufficient conditions using two real-world distribution circuits.

II. PROBLEM FORMULATION

Due to space limit, we introduce here an abridged version of the branch flow model; see, e.g., [1], [2] for more details.

A. Branch flow model for radial networks

TABLE I: Notations.

V_i, v_i	complex voltage on bus i with $v_i = \|V_i\|^2$
$s_i = p_i + \mathbf{i}q_i$	complex net load on bus i
I_{ij}, ℓ_{ij}	complex current from buses i to j with $\ell_{ij} = \|I_{ij}\|^2$
$S_{ij} = P_{ij} + \mathbf{i}Q_{ij}$	complex power flowing out from buses i to bus j
$z_{ij} = r_{ij} + \mathbf{i}x_{ij}$	impedance on line (i, j)

Consider a radial distribution circuit that consists of a set N of buses and a set E of distribution lines connecting these buses. We index the buses in N by $i = 0, 1, \ldots, n$, and denote a line in E by the pair (i, j) of buses it connects. Bus 0 represents the substation and other buses in N represent branch buses. For each line $(i, j) \in E$, let I_{ij} be the complex current flowing from buses i to j, $z_{ij} = r_{ij} + \mathbf{i} x_{ij}$ the impedance on line (i, j), and $S_{ij} = P_{ij} + \mathbf{i} Q_{ij}$ the complex power flowing from buses i to bus j. On each bus $i \in N$, let V_i be the complex voltage and s_i be the complex net load, i.e., the consumption minus generation. As customary, we assume that the complex voltage V_0 on the substation bus is given.

The branch flow model was first proposed in [3], [4] to model power flows in a steady state in a radial distribution circuit:

$$p_j = P_{ij} - r_{ij}\ell_{ij} - \sum_{k:(j,k)\in E} P_{jk}, \quad j = 1, \ldots, n \quad (1)$$

$$q_j = Q_{ij} - x_{ij}\ell_{ij} - \sum_{k:(j,k)\in E} Q_{jk}, \quad j = 1, \ldots, n \quad (2)$$

$$v_j = v_i - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (r_{ij}^2 + x_{ij}^2)\ell_{ij}, \\ (i, j) \in E \quad (3)$$

$$\ell_{ij} = \frac{P_{ij}^2 + Q_{ij}^2}{v_i}, \quad (i, j) \in E, \quad (4)$$

where $\ell_{ij} := |I_{ij}|^2$, $v_i := |V_i|^2$, and p_i and q_i are the real and reactive net loads at node i. Equations (1)–(4) define a system of equations in the variables $(P, Q, \ell, v) := (P_{ij}, Q_{ij}, \ell_{ij}, (i, j) \in E, \ v_i, i = 1, \ldots, n)$, which do not include phase angles of voltages and currents. Given an (P, Q, ℓ, v), these phase angles can be uniquely determined for radial networks. This is not the case for mesh networks; see [1] for exact conditions under which phase angles can be recovered for (an extension of the model here for) mesh networks.

B. Optimal power flow

Consider the problem of minimizing a cost function over the network where the optimization variables are $p := (p_1, \ldots, p_n)$, $q := (q_1, \ldots, q_n)$, as well as (P, Q, ℓ, v). Let

$$p_i := p_i^c - p_i^g, \quad q_i := q_i^c - q_i^g,$$

where p_i^c and q_i^c are the real and reactive power consumption at node i, and p_i^g and q_i^g are the real and reactive power generation at node i. In addition to power flow equations (1)–(4), we impose the following constraints on power consumption and generation:

$$\underline{p}_i^c \leq p_i^c \leq \overline{p}_i^c, \quad \underline{q}_i^c \leq q_i^c \leq \overline{q}_i^c, \quad i = 1, \ldots, n. \quad (5)$$

$$\underline{p}_i^g \leq p_i^g \leq \overline{p}_i^g, \quad \underline{q}_i^g \leq q_i^g \leq \overline{q}_i^g, \quad i = 1, \ldots, n. \quad (6)$$

$$f_i^c(p_i^c, q_i^c) \leq 0, \quad f_i^g(p_i^g, q_i^g) \leq 0, \quad i = 1, \ldots, n. \quad (7)$$

Here, equation (7) models additional constraints on (p_i^c, q_i^c) and (p_i^g, q_i^g). For example, for PV generators, $(p_i^g)^2 + (q_i^g)^2 \leq$

C^2 where C is the capacity of the PV generation [22]. We assume f_i^c, f_i^g are convex for all $i = 1, \cdots, n$.

Finally, the voltage magnitudes must be maintained to be above certain thresholds:

$$\underline{v}_i \leq v_i, \quad i = 1, \ldots, n. \quad (8)$$

Here we do not impose upper bounds on the voltage magnitudes. However, we derive below upper bounds on the *optimal* voltage magnitudes.

The objective of the optimal power flow problem is to minimize the power generation costs $C_i(p_i^g)$, the power losses $r_{i,j}\ell_{i,j}$, and maximize the user utilities $f_i(p_i^c)$:[1]

OPF:

$$\min_{P,Q,\ell,v,p,q} \quad \sum_{i=1}^{n} C_i(p_i^g) - \sum_{i=1}^{n} f_i(p_i^c) + \sum_{(i,j)\in E} r_{i,j}\ell_{i,j}$$

$$\text{s.t.} \quad (1) - (4), \ (5) - (8).$$

OPF is NP hard in general, due to the quadratic equality constraint (4).

III. EXACT RELAXATION

A. Second-order cone relaxation

Following [17], [1], [2], we relax the quadratic equalities in (4) into inequalities and consider the following convex relaxation of OPF.

ROPF:

$$\min_{P,Q,l,v,p,q} \quad \sum_{i=1}^{n} C_i(p_i^g) - \sum_{i=0}^{n} f_i(p_i^c) + \sum_{(i,j)\in E} r_{i,j}\ell_{i,j}$$

$$\text{s.t.} \quad (1) - (3), \ (5) - (8)$$

$$\ell_{ij} \geq \frac{P_{ij}^2 + Q_{ij}^2}{v_i}, \quad (i, j) \in E. \quad (9)$$

Obviously, ROPF provides a lower bound on OPF. It was shown in [17], [1] that this relaxation is exact when there are no upper bounds on the real and reactive power consumptions in (5) but with upper bounds on the voltage magnitudes in (8).

The main result of this chapter is a variety of sufficient conditions for exact relaxation when there are no upper bounds on the voltage magnitudes. Given a solution of the relaxed problem ROPF, one can always check if equality is attained in (4). If it is, then the relaxed solution is optimal for the original problem OPF as well. Otherwise, it is not feasible for OPF. Our goal is to develop sufficient conditions for exact relaxation that can be checked without having to solve ROPF first.

B. Sufficient condition for exact relaxation

We start by developing our results on a simple network, a one-line distribution circuit (main feeder). Then we will extend the results to general radial networks.

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[1]We can also include in the objective function the cost $C_0\left(\sum_{(0,j)\in E} P_{0,j}\right)$ on the total power fed into the radial network. This additional term does not change the results of the paper.

1) Line networks: For a one-line network, we can abbreviate r_{ij}, x_{ij}, P_{ij}, Q_{ij}, and l_{ij} by r_i, x_i, P_i, Q_i and l_i respectively, as shown in Figure 1. Rewrite the OPF problem
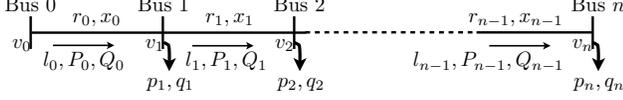

Fig. 1: A one-line distribution network.

in terms of the simplified notations as:

LOPF:

$$\min_{P,Q,\ell,v,p,q} \quad \sum_{i=1}^{n} C_i(p_i^g) - \sum_{i=1}^{n} f_i(p_i^c) + \sum_{i=0}^{n-1} r_i \ell_i$$

$$\text{s.t.} \quad \frac{P_i^2 + Q_i^2}{v_i} = \ell_i, i = 0, \cdots, n-1 \quad (10)$$

$$P_i = P_{i+1} + r_i \ell_i + p_{i+1}^c - p_{i+1}^g,$$
$$i = 0, \cdots, n-1 \quad (11)$$

$$Q_i = Q_{i+1} + x_i \ell_i + q_{i+1}^c - q_{i+1}^g,$$
$$i = 0, \cdots, n-1 \quad (12)$$

$$v_i - v_{i+1} = 2(r_i P_i + x_i Q_i) - (r_i^2 + x_i^2)\ell_i,$$
$$i = 0, \cdots, n-1 \quad (13)$$

$$(5) - (8).$$

The above optimization problem can be relaxed to the following second-order cone program:

RLOPF

$$\min_{P,Q,l,v,p,q} \quad \sum_{i=1}^{n} C_i(p_i^g) - \sum_{i=1}^{n} f_i(p_i^c) + \sum_{i=0}^{n-1} r_i \ell_i$$

$$s.t. \quad (5) - (8), (11) - (13)$$

$$\frac{P_i^2 + Q_i^2}{v_i} \leq \ell_i, \quad i = 0, \cdots, n-1. \quad (14)$$

The next lemma provides a sufficient condition guaranteeing that RLOPF is an exact relaxation of OPF. For each bus $i, k \in N \backslash \{0\}$, define

$$R_k := \sum_{j=0}^{k-1} r_j, \quad X_k := \sum_{j=0}^{k-1} x_j, \quad R_{i,k} := \sum_{j=i}^{k-1} r_j, \quad X_{i,k} := \sum_{j=i}^{k-1} x_j$$

as the cumulative resistance and reactance from the feeder or bus i to bus k. Also define $[a]^+ = \max(a, 0)$.

Lemma 1. *Any optimal solution* (P, Q, ℓ, v, p, q) *of RLOPF is also optimal for LOPF, provided that for each* $k \in N \backslash \{0\}$ *the following condition holds: if* $\frac{r_k}{x_k} - \frac{R_k}{X_k} \geq 0$, *then*

$$v_i + 2P_i \left(\frac{r_k}{x_k} X_k - R_{i,k} \right) + 2Q_i X_i > 0; \forall i < k \quad (15)$$

otherwise,

$$v_i + 2P_i R_i + 2Q_i \left(\frac{x_k}{r_k} R_k - X_{i,k} \right) > 0, \forall i < k \quad (16)$$

Moreover, for each node $i \in N \backslash \{0\}$, *the voltage is upper-bounded by:*[2]

$$v_i \leq v_0 - 2 \sum_{k=0}^{i-1} (r_k(P_k - r_k \ell_k) + x_k(Q_k - x_k \ell_k)).$$

Proof. Introducing dual variables for Problem LOPF, denoted as

$$\alpha = (\lambda_i, \theta_i, w_i, \xi_i, \gamma_i^c, \gamma_i^g, \bar{\zeta}_i^c, \underline{\zeta}_i^c, \bar{\eta}_i^c, \underline{\eta}_i^c, \bar{\zeta}_i^g, \underline{\zeta}_i^g, \bar{\eta}_i^g, \underline{\eta}_i^g, \mu_i),$$

where $\mu_i \geq 0$, $\xi_i \leq 0$. Let x denote the primal variables. The Lagrangian dual function is given as:

$$L(x, \alpha) = -\sum_{i=0}^{n} f_i(p_i^c) + \sum_{i=1}^{n} C_i(p_i^g) + \sum_{i=0}^{n-1} r_i \ell_i$$
$$+ \sum_{i=0}^{n-1} \lambda_i (P_i - P_{i+1} - r_i \ell_i - p_{i+1})$$
$$+ \sum_{i=0}^{n-1} \theta_i (Q_i - Q_{i+1} - x_i \ell_i - q_{i+1})$$
$$+ \sum_{i=0}^{n-1} \mu_i \left(\frac{P_i^2 + Q_i^2}{v_i} - \ell_i \right)$$
$$+ \sum_{i=0}^{n-1} w_i (v_i - v_{i+1} - 2(r_i P_i + x_i Q_i) + (r_i^2 + x_i^2)\ell_i)$$
$$+ \sum_{i=1}^{n} (\gamma_i^c f_i^c(p_i^c, q_i^c) + \gamma_i^g f_i^g(p_i^g, q_i^g))$$
$$+ \sum_{i=1}^{n} (\xi_i v_i + (\bar{\zeta}_i^c - \underline{\zeta}_i^c)p_i^c + (\bar{\eta}_i^c - \underline{\eta}_i^c)q_i^c)$$
$$+ \sum_{i=1}^{n} ((\bar{\zeta}_i^g - \underline{\zeta}_i^g)p_i^g + (\bar{\eta}_i^g - \underline{\eta}_i^g)q_i^g)$$
$$+ \sum_{i=1}^{n} (-\xi_i \underline{v}_i - \bar{p}_i^c \bar{\zeta}_i^c + \underline{p}_i^c \underline{\zeta}_i^c - \bar{q}_i^c \bar{\eta}_i^c)$$
$$+ \sum_{i=1}^{n} (\underline{q}_i^g \underline{\eta}_i^g - \bar{p}_i^g \bar{\zeta}_i^g + \underline{p}_i^g \underline{\zeta}_i^g - \bar{q}_i^g \bar{\eta}_i^g + \underline{q}_i^g \underline{\eta}_i^g).$$

If (x^*, α^*) are the primal-dual optimal, then $\frac{\partial L(x^*, \alpha^*)}{x} = 0$. In the following, we will drop the '$*$' for simplicity. Define $\beta_i = \frac{\mu_i}{v_i}$ for each $i = 0, \cdots, n-1$. $\frac{\partial L(x^*, \alpha^*)}{x} = 0$ gives that for all $i = 0, \cdots, n-1$,

$$\frac{\partial L}{P_i} = -\lambda_{i-1} + \lambda_i + 2\beta_i P_i - 2w_i r_i = 0, \quad (17)$$

$$\frac{\partial L}{Q_i} = -\theta_{i-1} + \theta_i + 2\beta_i Q_i - 2w_i x_i = 0, \quad (18)$$

$$\frac{\partial L}{\ell_i} = r_i - \lambda_i r_i - \theta_i x_i - \beta_i v_i + w_i(r_i^2 + x_i^2) = 0, \quad (19)$$

and for all $i = 1, \cdots, n$,

$$\frac{\partial L}{v_i} = -\mu_i \frac{P_i^2 + Q_i^2}{v_i^2} - w_{i-1} + w_i + \xi_i = 0, \quad (20)$$

[2]Note that $P_k - r_k \ell_k$ and $Q_k - x_k \ell_k$ are the real and reactive power received by bus $k + 1$ from bus k.

where we introduce dummy variables $\lambda_{-1} = \theta_{-1} = \beta_n = \mu_n = l_n = w_n = 0$.

By (20), it is straightforward to show that $w_i \leq 0$, for $i = 0, \cdots, n$. Note that for $i = n$, we have $-w_{n-1} + \xi_n = 0$, which implies that $w_{n-1} = \xi_n \leq 0$. Then by induction, we can show that $w_i \leq 0$, for $i = 0, \cdots, n-1$.

The rest of the proof will show that $\beta_i > 0$ for all $i \in N$. Then by complementary slackness, we know that all equality (10) holds for each $i \in N$, implying that exact relaxation holds.

Suppose there exists such $i \in N$ that $\beta_i = 0$. Let $k := \min\{i \in N : \beta_i > 0\}$. Suppose $k = 0$. By (17,18,19), we have:

$$\lambda_0 - 2w_0 r_0 = 0 \qquad (21)$$
$$\theta_0 - 2w_0 x_0 = 0 \qquad (22)$$
$$r_0 - \lambda_0 r_0 - \theta_0 x_0 + w_0(r_0^2 + x_0^2) = 0. \qquad (23)$$

Substituting (21,22) into (23), we have:

$$r_0 - w_0(r_0^2 + x_0^2) = 0.$$

The LHS is strictly positive since $r_0 > 0$ and $w_0 \leq 0$. Therefore by contradiction, $k > 0$.

Suppose $k > 0$. Write λ_l and θ_l for each $l \leq k$ in terms of w and β. Summing up (17) from $i = 0$ to $i = l$, we have:

$$\lambda_l = 2\left(\sum_{i=0}^{l} w_i r_i - \sum_{i=0}^{l} \beta_i P_i\right), l = 0, \cdots, k. \qquad (24)$$

Similarly, summing up (18) from $i = 0$ to $i = l$:

$$\theta_l = 2\left(\sum_{i=0}^{l} w_i x_i - \sum_{i=0}^{l} \beta_i Q_i\right), l = 0, \cdots, k. \qquad (25)$$

Substituting (24,25) into (19) for $l \leq k$:

$$\beta_l v_l = r_l + w_l(r_l^2 + x_l^2) - 2r_l\left(\sum_{i=0}^{l} w_i r_i - \sum_{i=0}^{l} \beta_i P_i\right)$$
$$-2x_l\left(\sum_{i=0}^{l} w_i x_i - \sum_{i=0}^{l} \beta_i Q_i\right). \qquad (26)$$

Summing up (26) from $l = 0$ to $l = k - 1$ gives:

$$\sum_{l=0}^{k-1} \beta_l v_l = R_k + \sum_{l=0}^{k-1} w_l(r_l^2 + x_l^2)$$
$$-2\sum_{l=0}^{k-1}\sum_{i=0}^{l}(w_i r_i r_l - \beta_i P_i r_l + w_i x_i x_l - \beta_i Q_i x_l)$$
$$= R_k + \sum_{l=0}^{k-1} w_l(r_l^2 + x_l^2)$$
$$-2\sum_{i=0}^{k-1}\sum_{l=i}^{k-1}(w_i r_i r_l - \beta_i P_i r_l + w_i x_i x_l - \beta_i Q_i x_l)$$
$$= R_k - 2\sum_{i=0}^{k-1}(w_i r_i R_{i,k} - \beta_i P_i R_{i,k})$$
$$-2\sum_{i=0}^{k-1}(w_i x_i X_{i,k} - \beta_i Q_i X_{i,k}) + \sum_{l=0}^{k-1} w_l\left(r_l^2 + x_l^2\right). \qquad (27)$$

where $R_{i,k} := \sum_{l=i}^{k-1} r_l$ and $X_{i,k} := \sum_{l=i}^{k-1} x_l$.

Also, by (26) with $l = k$, we have:

$$0 = r_k - 2r_k \sum_{i=0}^{k-1} w_i r_i - 2x_k \sum_{i=0}^{k-1} w_i x_i$$
$$-w_k(r_k^2 + x_k^2) + 2r_k\left(\sum_{i=0}^{k-1} \beta_i P_i\right) + 2x_k\left(\sum_{i=0}^{k-1} \beta_i Q_i\right) \qquad (28)$$

If $\frac{r_k}{x_k} - \frac{R_k}{X_k} \geq 0$, applying the following operation on (27) and (28), $\frac{(28)}{x_k} - \frac{(27)}{X_k}$, and using the fact that $w_i \leq 0$ for all $i \in N$,

$$\frac{r_k}{x_k} - \frac{R_k}{X_k} - 2\sum_{i=0}^{k-1} w_i r_i\left(\frac{r_k}{x_k} - \frac{R_{i,k}}{X_k}\right)$$
$$-2\sum_{i=0}^{k-1} w_i x_i\left(1 - \frac{X_{i,k}}{X_k}\right)$$
$$\leq -\sum_{i=0}^{k-1} \frac{\beta_i}{X_k}\left(v_i + 2P_i\left(\frac{r_k}{x_k}X_k - R_{i,k}\right) + 2Q_i X_i\right). \qquad (29)$$

Note that under the condition specified in Section III, the RHS of inequality (29) is negative; but the LHS is non-negative. Therefore we have a contradiction.

Otherwise, if $\frac{r_k}{x_k} - \frac{R_k}{X_k} \leq 0$, applying the following operation on (27) and (28), $\frac{(28)}{r_k} - \frac{(27)}{R_k}$, and following similar arguments, we will get a contradiction as well. We can therefore claim the sufficient condition in the Lemma.

In the rest of the proof we will bound the voltage on each node $i \in N \backslash \{0\}$. Equation (13) implies that for each $k \in N$,

$$v_{k+1} = v_k - 2(r_k P_k + x_k Q_k) + (r_k^2 + x_k^2)\ell_k$$
$$\leq v_k - 2(r_k P_k + x_k Q_k) + 2(r_k^2 + x_k^2)\ell_k$$
$$= v_k - 2(r_k(P_k - r_k\ell_k) + x_k(Q_k - x_k\ell_k)) .$$

Summing up this inequality from $k = 0$ to $k = i - 1$, we have

$$v_i \leq v_0 - 2\sum_{k=0}^{i-1}(r_k(P_k - r_k\ell_k) + x_k(Q_k - x_k\ell_k)).$$

$\square$

The condition in Lemma 1 is not checkable before solving RLOPF as it involves a solution (P, Q, ℓ, v, p, q) of RLOPF. We now provide a checkable condition by bounding v_i, P_i, Q_i in terms of system parameters $\underline{p}_i, \bar{p}_i, \underline{q}_i, \bar{q}_i, \underline{v}_i$. Define

$$\underline{P}_i^{nom} \triangleq \sum_{j=i+1}^{n}(\underline{p}_j^c - \bar{p}_j^g); \quad \underline{Q}_i^{nom} \triangleq \sum_{j=i+1}^{n}(\underline{q}_j^c - \bar{q}_j^g).$$

Using (11–12) we can iteratively derive that for any $i \in N$,

$$P_i \geq P_i - r_i\ell_i \geq \underline{P}_i^{nom},$$

and

$$Q_i \geq Q_i - r_i\ell_i \geq \underline{Q}_i^{nom}.$$

Combining the above two inequalities with Lemma 1 gives the following result.

Theorem 2. *Any optimal solution of RLOPF is also optimal for LOPF, provided that for each $k \in N \backslash \{0\}$ the following condition holds: if $\frac{r_k}{x_k} - \frac{R_k}{X_k} \geq 0$,*

$$\underline{v}_i + 2\underline{P}_i^{nom}\left(\frac{r_k}{x_k}X_k - R_{i,k}\right) + 2\underline{Q}_i^{nom}X_i > 0; \forall i < k; \quad (30)$$

otherwise,

$$\underline{v}_i + 2\underline{P}_i^{nom}R_i + 2\underline{Q}_i^{nom}\left(\frac{x_k}{r_k}R_k - X_{i,k}\right) > 0, \forall i < k . \quad (31)$$

Moreover, for each node $i \in N \backslash \{0\}$, the voltage is upper-bounded by:

$$v_i \leq v_0 - 2\sum_{k=0}^{i-1}\left(r_k\underline{P}_k^{nom} + x_k\underline{Q}_k^{nom}\right).$$

Since $\underline{v}_i > 0$ for each $i \in N$, we have the following special case. If $\underline{P}_i^{nom} > 0$ and $\underline{Q}_i^{nom} > 0$ for all $i \in N \backslash \{0\}$, then the right-hand sides of (30,31) are always non-positive, which implies that the sufficient condition in Theorem 2 is always satisfied. Hence the relaxation is exact provided that both the real and reactive powers do not flow backward. This condition in the special case is more stringent than (30,31) and usually does not hold in practice. The sufficient condition (30,31) depends only on how $\underline{v}_i$ compare with the products of resistances (reactances) and real (reactive) powers. In practice, $|V| \sim 1$ p.u., $r, x \ll 1$ p.u., $\frac{r}{x} \sim [0.1, 10]$, and $p, q < 1$ p.u.. As we show in Section IV, condition (30,31) usually holds when the system parameters are in these ranges.

2) General radial networks: We now extend Lemma 1 and Theorem 2 to general radial distribution circuits. Given a radial network:

- For each node i, k, denote the unique path from i to node k by $\mathcal{P}_{i,k} \triangleq \{(j_1, j_2) : (j_1, j_2) \in E$ is on the path from node i to node $k\}$.
- Define the cumulative resistance and reactance from root i to node k as $R_{i,k} \triangleq \sum_{(j_1,j_2)\in\mathcal{P}_{i,k}} r_{j_1,j_2}$ and $X_{i,k} \triangleq \sum_{(j_1,j_2)\in\mathcal{P}_{i,k}} x_{j_1,j_2}$.

It is straightforward to extend Lemma 1 to the case of general radial networks.

Lemma 3. *Any optimal solution (P, Q, ℓ, v, p, q) of ROPF is also optimal for OPF, provided that for each $(k, l) \in E$ the following condition holds: if $\frac{r_{k,l}}{x_{k,l}} - \frac{R_{0,k}}{X_{0,k}} \geq 0$, then for any $(i, j) \in \mathcal{P}_k$,*

$$v_i + 2P_{i,j}\left(\frac{r_{k,l}}{x_{k,l}}X_{0,k} - R_{i,k}\right) + 2Q_{i,j}X_{0,i} > 0; ; \quad (32)$$

otherwise, for any $(i, j) \in \mathcal{P}_k$,

$$v_i + 2P_{i,j}R_{0,i} + 2Q_{i,j}\left(\frac{x_{k,l}}{r_{k,l}}R_{0,k} - X_{i,k}\right) > 0. \quad (33)$$

Moreover, for each node $i \in N \backslash \{0\}$, the voltage is upper-bounded by:

$$v_i \leq v_0 - 2\sum_{(j,k)\in\mathcal{P}_i}\left(r_{j,k}(P_{j,k} - r_{j,k}\ell_{j,k}) + x_{j,k}(Q_{j,k} - x_{i,j}\ell_{j,k})\right)$$

Proof. The proof for Lemma 1 can be easily extended to a radial network. We can first prove $w_{i,j} \leq 0$ for each $(i, j) \in E$ by induction (from leaves to the feeder.). To prove $\beta_{i,j} > 0$, just focus on each lateral and use the same arguments as those for a line distribution network to prove that $\beta_{i,j} > 0$ on each lateral. $\square$

Similarly, this lemma involves a solution $v_i, P_{i,j}, Q_{i,j}$ of ROPF. For a sufficient condition that does not require solving ROPF first, define

$$\underline{P}_j^{nom} \triangleq \sum_{i\in D(j)}^{n}(\underline{p}_i^c - \bar{p}_i^g); \quad \underline{Q}_j^{nom} \triangleq \sum_{i\in D(j)}^{n}(\underline{q}_i^c - \bar{q}_i^g),$$

for each $j = 1, \cdots, n$. Here $D(j)$ is the set of all the descendants of j including j itself.[3] Note that for any $(i, j) \in E$,

$$P_{i,j} \geq P_{i,j} - r_{i,j}\ell_{i,j} \geq \underline{P}_j^{nom}, Q_{i,j} \geq Q_{i,j} - x_{i,j}\ell_{i,j} \geq \underline{Q}_j^{nom}.$$

Lemma 3 then implies the following extension of Theorem 2.

Theorem 4. *Any optimal solution of ROPF is also optimal for OPF, provided that for each $(k, l) \in E$ the following condition holds: if $\frac{r_{k,l}}{x_{k,l}} - \frac{R_{0,k}}{X_{0,k}} \geq 0$, then for any $(i, j) \in \mathcal{P}_k$,*

$$\underline{v}_i + 2\underline{P}_j^{nom}\left(\frac{r_{k,l}}{x_{k,l}}X_{0,k} - R_{i,k}\right) + 2\underline{Q}_j^{nom}X_{0,i} > 0; \quad (34)$$

otherwise, for any $(i, j) \in \mathcal{P}_k$,

$$\underline{v}_i + 2\underline{P}_j^{nom}R_{0,i} + 2\underline{Q}_j^{nom}\left(\frac{x_{k,l}}{r_{k,l}}R_{0,k} - X_{i,k}\right) > 0. \quad (35)$$

Moreover, for each node $i \in N \backslash \{0\}$, the voltage is upper-bounded by:

$$v_i \leq v_0 - 2\sum_{(j,k)\in\mathcal{P}_i}\left(r_{j,k}\underline{P}_k^{nom} + x_{j,k}\underline{Q}_k^{nom}\right).$$

Since $|V| \sim 1$ p.u., $r, x \ll 1$ p.u., and $p, q < 1$ p.u. in practice, the condition in Theorem 4 holds for both a 47-bus distribution circuit and a 56-bus distribution circuit of Southern California Edison (SCE), as show in Section IV.

IV. CASE STUDY

In this section we evaluate these conditions for exact relaxation for two distribution circuits of SCE with high penetration of photovoltaic (PV) generation [17], [23]. Figures 2 and 3 show a 47-bus and a 56-bus distribution circuit respectively and Table II and III list the corresponding network data including line impedances, peak MVA demand of loads, and the nameplate capacity of the shunt capacitors and the photovoltaic generations. Note that in the 47-bus circuit, bus 1 indicates the substation, and there are 5 photovoltaic (PV) generators located on buses 13, 17, 19, 23 and 24. In the 56-bus circuit, there is 1 photovoltaic (PV) generator located on bus 45.

[3] A rigorous definition of $D(j)$ is: $D(j) \triangleq \{l \in N :$ there exist a sequence of nodes, $j_0, j_1, \cdots, j_m$, such that $j_0 = j, j_m = l$, and $(j_i, j_{i+1}) \in E, \forall i = 0, \ldots, m-1$, where $m \geq 0\}$.
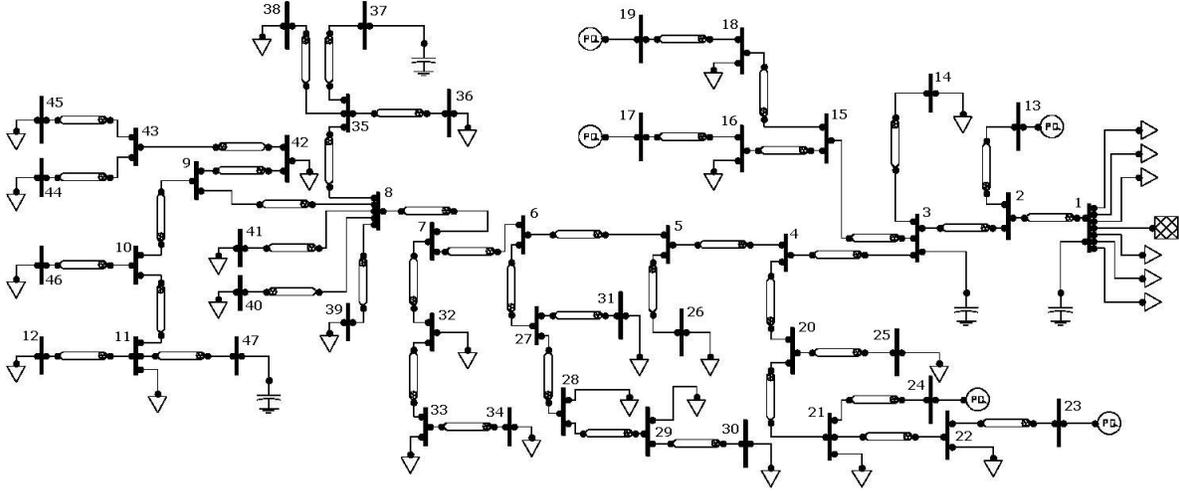
Fig. 2: Schematic diagram of a 47-bus SCE distribution systems.

TABLE II: Line impedances, peak spot load KVA, Capacitors and PV generation's nameplate ratings for the distribution circuit in Figure 2.

| Network Data															
Line Data				Line Data				Line Data				Load Data		Load Data		PV Generators	
From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	Bus No.	Peak MVA	Bus No.	Peak MVA	Bus No.	Nameplate Capacity
1	2	0.259	0.808	8	41	0.107	0.031	21	22	0.198	0.046	1	30	34	0.2		
2	13	0	0	8	35	0.076	0.015	22	23	0	0	11	0.67	36	0.27	13	1.5MW
2	3	0.031	0.092	8	9	0.031	0.031	27	31	0.046	0.015	12	0.45	38	0.45	17	0.4MW
3	4	0.046	0.092	9	10	0.015	0.015	27	28	0.107	0.031	14	0.89	39	1.34	19	1.5 MW
3	14	0.092	0.031	9	42	0.153	0.046	28	29	0.107	0.031	16	0.07	40	0.13	23	1 MW
3	15	0.214	0.046	10	11	0.107	0.076	29	30	0.061	0.015	18	0.67	41	0.67	24	2 MW
4	20	0.336	0.061	10	46	0.229	0.122	32	33	0.046	0.015	21	0.45	42	0.13		
4	5	0.107	0.183	11	47	0.031	0.015	33	34	0.031	0	22	2.23	44	0.45	Shunt Capacitors	
5	26	0.061	0.015	11	12	0.076	0.046	35	36	0.076	0.015	25	0.45	45	0.2	Bus No.	Nameplate Capacity
5	6	0.015	0.031	15	18	0.046	0.015	35	37	0.076	0.046	26	0.2	46	0.45		
6	27	0.168	0.061	15	16	0.107	0.015	35	38	0.107	0.015	28	0.13			1	6000 KVAR
6	7	0.031	0.046	16	17	0	0	42	43	0.061	0.015	29	0.13	V_{base} = 12.35kV		3	1200 KVAR
7	32	0.076	0.015	18	19	0	0	43	44	0.061	0.015	30	0.2			37	1800 KVAR
7	8	0.015	0.015	20	21	0.122	0.092	43	45	0.061	0.015	31	0.07			47	1800 KVAR
8	40	0.046	0.015	20	25	0.214	0.046					32	0.13				
8	39	0.244	0.046	21	24	0	0					33	0.27				
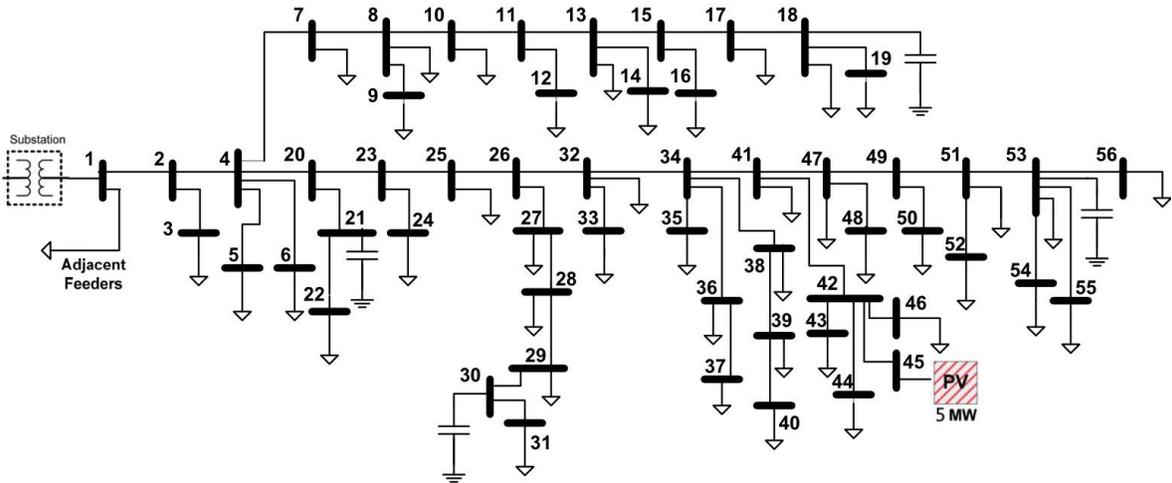
Fig. 3: Schematic diagram of a 56-bus SCE distribution systems.

A. Verifying sufficient conditions

We verify that the condition in Theorem 4 holds in both circuits. To calculate $\underline{P}^{nom}$ and $\underline{Q}^{nom}$, we only need values

for lower bounds of (p_i^c, q_i^c) and upper bounds of (p_i^g, q_i^g):

• For load buses, we set $\underline{p}_i^c$ to be 0 and set $\underline{q}_i^c$ to be the

TABLE III: Line impedances, peak spot load kVA, capacitors and PV generation's nameplate ratings for the distribution circuit in Figure 3.

Network Data																	
Line Data				Line Data				Line Data				Load Data		Load Data		Load Data	
From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	Bus No.	Peak MVA	Bus No.	Peak MVA	Bus No.	Peak MVA
1	2	0.160	0.388	20	21	0.251	0.096	39	40	2.349	0.964	3	0.057	29	0.044	52	0.315
2	3	0.824	0.315	21	22	1.818	0.695	34	41	0.115	0.278	5	0.121	31	0.053	54	0.061
2	4	0.144	0.349	20	23	0.225	0.542	41	42	0.159	0.384	6	0.049	32	0.223	55	0.055
4	5	1.026	0.421	23	24	0.127	0.028	42	43	0.934	0.383	7	0.053	33	0.123	56	0.130
4	6	0.741	0.466	23	25	0.284	0.687	42	44	0.506	0.163	8	0.047	34	0.067	Shunt Cap	
4	7	0.528	0.468	25	26	0.171	0.414	42	45	0.095	0.195	9	0.068	35	0.094	Bus	Mvar
7	8	0.358	0.314	26	27	0.414	0.386	42	46	1.915	0.769	10	0.048	36	0.097	19	0.6
8	9	2.032	0.798	27	28	0.210	0.196	41	47	0.157	0.379	11	0.067	37	0.281	21	0.6
8	10	0.502	0.441	28	29	0.395	0.369	47	48	1.641	0.670	12	0.094	38	0.117	30	0.6
10	11	0.372	0.327	29	30	0.248	0.232	47	49	0.081	0.196	14	0.057	39	0.131	53	0.6
11	12	1.431	0.999	30	31	0.279	0.260	49	50	1.727	0.709	16	0.053	40	0.030	Photovoltaic	
11	13	0.429	0.377	26	32	0.205	0.495	49	51	0.112	0.270	17	0.057	41	0.046	Bus	Capacity
13	14	0.671	0.257	32	33	0.263	0.073	51	52	0.674	0.275	18	0.112	42	0.054		
13	15	0.457	0.401	32	34	0.071	0.171	51	53	0.070	0.170	19	0.087	43	0.083	45	5MW
15	16	1.008	0.385	34	35	0.625	0.273	53	54	2.041	0.780	22	0.063	44	0.057		
15	17	0.153	0.134	34	36	0.510	0.209	53	55	0.813	0.334	24	0.135	46	0.134	V_{base} = 12kV	
17	18	0.971	0.722	36	37	2.018	0.829	53	56	0.141	0.340	25	0.100	47	0.045		
18	19	1.885	0.721	34	38	1.062	0.406					27	0.048	48	0.196		
4	20	0.138	0.334	38	39	0.610	0.238					28	0.038	50	0.045		

negative of peak MVA value.

- For PV generators, we set $\bar{p}_i^g$ to be the generators' capacities.
- For shunt capacitors, we treat them as reactive power generators and set $\bar{q}_i^g$ to be their shunt capacities.

After checking conditions in Theorem 4, we know that as long as the voltage magnitudes are maintained above 85% of the nominal value (which they are in practice) for each bus i, the conditions holds and ROPF is an exact relaxation of OPF.

Remark 1. *All the above analysis is worst-case. In reality, $\underline{p}_i^c$ and $\underline{q}_i^c$ tend to be larger than the values we used above, and $\underline{p}_i^g$ and $\underline{q}_i^g$ smaller. This implies larger $(\underline{P}^{nom}, \underline{Q}^{nom})$ and larger values for the left-hand side of inequality (34) and of inequality (35) than the values we have calculated above. Thus the sufficient condition in Corollary 4 is easier to meet in practice.*

Remark 2. *The condition in Theorem 4 can be used as a rule of thumb for designing distribution circuits that will ensure that ROPF is an exact relaxation of OPF. Specifically, if the distribution lines have smaller resistance and reactance, then condition in Theorem 4 is easier to satisfy.*

B. Simulation

We have also solved Problem ROPF using the CVX toolbox [24]. In the simulation:

- for each load bus, we set $\bar{p}_i^c$ and $\bar{q}_i^c$ as the peak MVA value, and $\underline{p}_i^c$, $\underline{q}_i^c$ as half of the peak MVA value. We use utility functions of the form $-a_i(p_i - \bar{p}_i^c)^2 + b_i$ where a_i, b_i are drawn randomly from $[2.5, 5]$;
- for each PV generator, we set $(\bar{p}_i^g, \bar{q}_i^g)$ as the generator's capacity value, and $\underline{p}_i^g$, $\underline{q}_i^g$ as 0. We use cost functions of the form $a_i p_i^2$ where a_i are drawn randomly from $[2.5, 5]$;
- for each shunt capacitor, we treat them as reactive power generators and set $\bar{p}_i^c = \underline{p}_i^c = \underline{q}_i^c = 0$ and $\bar{q}_i^g$ as their shunt capacities;

- we use a cost $C_0(P_0) := C_0(\sum_{j:(0,j)\in E} P_{0,j})$ on the total power fed into this distribution system of the form $a_0 P_0^2 + b_0 P_0$ with $a_0 = 0.1, b_0 = 0.1$.

After solving ROPF using the CVX toolbox for both the 47-bus system and the 56-bus system, we verify that the solutions of ROPF satisfy the equality constraint (4) and are therefore optimal for OPF. This implies that ROPF indeed is an exact relaxation of OPF for both distribution circuits. Moreover, in each case, the maximum voltage magnitude of the optimal solution is 12.35KV which is the voltage magnitude of feeder and much less than the upper bound we provided in Section IV-A.

V. Conclusion

We have studied the second-order cone relaxation of the optimal power flow problem in radial networks using the branch flow model. We provide sufficient conditions under which the relaxation is exact when there are no upper bounds on the voltage magnitudes. We have verified that these conditions are satisfied in two real-world distribution circuits.

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