A Market Mechanism for Electric Distribution Networks

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***Abstract*— To encourage participation of end-users in transforming the power grid into a distributed, autonomous, adaptive and resilient grid, an efficient electricity market, especially in distribution networks, plays an important role. However, the externalities associated with the power flow and network operating constraints constitute a significant barrier to form efficient markets. Traditionally, there is a central regulator to determine locational marginal prices in order to compensate the externalities. In this paper, we present a market mechanism for a radial distribution network which internalize the externalities within private decisions through defining trading rules that efficiently allocate the externalities to individuals in the network. Specifically, we focus on the external costs associated with voltage constraints and line losses. We show that a competitive market could be established for distribution services and electricity to achieve a social optimum within a power pool.**

I. Introduction

To transfer the power grid into a distributed, autonomous, adaptive, and resilient grid requires great participation from end-users, such as demand response. To this end, it involves sophisticated design of the energy management systems, incentives, and a viable trading market in order to guarantee the grid's efficiency and reliability. Among those, pricing policies especially in distribution networks play a fundamental role in shaping consumers behavior. According to Kirchoff's law, any local change (e.g. power injection) in the power network will affect the whole power network state (e.g. either power flow or voltage or both). As a result, the local decisions of users usually cause external costs associated with power flows, voltages, line losses, etc. These external costs constitute a significant barrier to form efficient markets for electricity networks.

There have been many literature studying different pricing schemes either for wholesale markets or retail markets, either day-ahead or real-time, either to smooth the demand profile over time or to call on users' action to compensate the fluctuations of renewable energy [1], [2], [3], [4], [5], [6], [7], [8], [9]. However, most of the work either omit the power network structure or use the DC power flow model. Thus the pricing policies could not reflect the externalities related to the AC power networks. Recently there have been work applying AC optimal power model and propose locational marginal price [5], [6], [9]. Though the locational marginal price is an efficient pricing scheme to promote social welfare, the issues are that: i) it requires a central regulator to gather the whole system information and/or to coordinate consumers in order to determine an efficient locational marginal price which compensate the externalities, ii) there is a lack of deep understanding of how the externalities are reflected in the locational marginal prices, or in other words, how the externalities are allocated to individuals in the network through the locational prices. As a result, it remains an open question how to design an decentralized market to allow individuals trading electricity/power with each other without affecting the global network performance.

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In this paper we present a market mechanism for a radial distribution network which internalizes the external costs within private decisions, such as local trading transactions. We define trading rules which efficiently allocate the external costs to individuals in the network. [10] presents a market design for transmission system where externalities of transmission line congestion and transmission losses play a central role. However, in distribution networks where most end-users are, voltage capacity constraints (which can be interpreted as another type of congestion) play a more significant role than transmission line congestion because it affects the power delivery and power quality. Therefore in this paper we focus on the externality cost associated with voltage capacity constraints and distribution line losses. The main idea of this paper is similar to the work of [10] where a market for transmission network is proposed. Specifically, we adopt the notion of voltage capacity rights and introduce a trading rule that specifies the voltage rights and distribution line loss compensation required for each transaction of power dispatch. The trading rule takes into account the effects of electricity transactions on the voltage and distribution line losses. With these property rights and the trading rules, we demonstrate that a competitive equilibrium can achieve a social optimum. The proposed mechanism serves as an initial step to provide guidelines for creating an efficient and decentralized market for electric distribution services.

The following of the paper is organized as follows: Section II present an AC power flow model for distribution network that we will use in this paper; Section III present a simple example with only 3 node to illustrate the idea of how voltage capacity constraints will shape electricity market; Section IV provide a market design and trading rule considering the voltage constraints by neglecting the power losses; Section V extend the idea to the full AC network model by considering the power losses; and Section VII discusses the limitation of the results and future work and concludes the paper.

II. Network Model

In this section, we advocate the use of branch flow model for a distribution network. In contrast to bus injection models which focus on nodal variables such as bus current and power injections, branch flow models focus on currents and power flows on individual branches [11]. They have been used mainly for modeling distribution circuits which tend to be radial. In this paper, we focus on radial distribution networks.

A. Branch flow model for radial distribution network

Consider a radial distribution circuit that consists of a set N of buses and a set E of distribution lines connecting these buses. We index the buses in N by $i = 0, 1, \ldots, n$, where bus 0 represents the substation (or the feeder) and other buses in N represent branch buses. We also denote a line in E by the pair (i, j) of buses it connects where j is closer to the feeder 0. We call j is the parent of i, denoted by $\pi(i)$, and i is the child of j. Denote the child set of j as $\delta(j) := \{i : (i, j) \in E\}$. Thus a link (i, j) can be denoted as $(i, \pi(i))$. We also denote the unique path from node 0 to node i as $\mathcal{P}_i$, i.e. $\mathcal{P}_i := \{k : k \text{ is on the path from node 0 to node } i\}$, and the descent set of node i as Δ_i, i.e., $\Delta_i := \{k : i \in \mathcal{P}_k\}$. We let $i \in \mathcal{P}_i$, $0 \notin \mathcal{P}_i$, $i \in \Delta_i$.

For each line $(i, \pi(i)) \in E$, let I_i be the complex current flowing from buses i to $\pi(i)$, $z_i = r_i + \mathbf{i}x_i$ the impedance on the link, and $S_i = P_i + \mathbf{i}Q_i$ the complex power flowing from buses i to bus $\pi(i)$. On each bus $i \in N$, let V_i be the complex voltage, $s_i^d = p_i^d + \mathbf{i}q_i^d$, $s_i^g = p_i^g + \mathbf{i}q_i^g$ be the complex power consumption and power generation, and $s_i := p_i + \mathbf{i}q_i := s_i^g - s_i^d$ be the net power injection. As customary, we assume that the complex voltage V_0 on the substation bus is given and fixed. Define $\ell := |I_{ij}|^2$, $v_i := |V_i|^2$. Figure 1 shows a line distribution network explaining the notations.

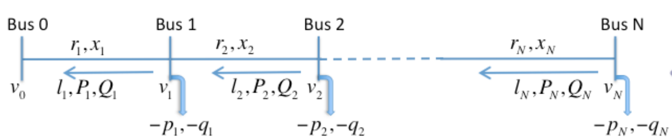

Fig. 1. A line distribution network with notations.

The branch flow model, first proposed in [12], [13] model power flows in a steady state in a radial distribution network:

$$S_i = s_i^g - s_i^d + \sum_{k \in \delta(i)} \left(S_k - (r_k + \mathbf{i}x_k)\ell_i\right), i = 0, \ldots, n \quad (1a)$$

$$v_i = v_{\pi(i)} + 2(r_i P_i + x_i Q_i) - (r_i^2 + x_i^2)\ell_i, \quad (1b)$$
$$i = 1, \ldots, n,$$

$$\ell_i = \frac{P_i^2 + Q_i^2}{v_i}, i = 1, \ldots, n \quad (1c)$$

where $S_0 := 0 + \mathbf{i}0$. Notice that $s_0^g - s_0^d$ can be interpreted as the total power injection into the distribution network from the main grid through the feeder bus 0. Equations (1) define a system of equations in the variables $(P, Q, \ell, v) := (P_{ij}, Q_{ij}, \ell_{ij}, (i, j) \in E, i = 1, \ldots, n)$, which do not include phase angles of voltages and currents. Given an (P, Q, ℓ, v) these phase angles can be uniquely determined for radial networks [14].

In addition to power flow equations (1), voltage magnitudes must be maintained within certain operating constraints:

$$\underline{v}_i \le v_i \le \bar{v}_i, i = 1, \ldots, n. \quad (2)$$

B. Social welfare maximization–demand response objective

The objective of the demand response considered in this paper is to maximize the social welfare–minimize the power generation costs $C_i(p_i^g)$, the power losses $r_{ij}\ell_{ij}$, and maximize the user utilities $B_i(p_i^c)$:[1]

$$\begin{aligned}
\min \quad & \sum_{i=0}^{n} C_i(p_i^g) - \sum_{i=0}^{n} B_i(p_i^c) + \sum_{(i,j)\in E} r_{ij}\ell_{ij} \quad (3)\\
\text{s.t.} \quad & (1), (2)\\
\text{over} \quad & p_i^g \in [\underline{p}_i^g, \bar{p}_i^g], p_i^d \in [\underline{p}_i^d, \bar{p}_i^d], i = 0, \ldots, n\\
& P_i, Q_i, \ell_i, v_i, i = 1, \ldots, n
\end{aligned}$$

C. Linear approximation of the network model

Real distribution circuits usually have very small r, x, i.e. $r, x << 1$, while $v \sim 1$. Thus real and reactive power losses are typically much smaller than power flows P_{ij}, Q_{ij}. Following[15], we neglect the higher order real and reactive power loss terms in (1) by setting $\ell_{ij} = 0$ and approximate P, Q, v using the following linear approximation, known as Simplified Distflow introduced in [15].

$$S_i = s_i^g - s_i^d + \sum_{k \in \delta(i)} S_k, i = 0, \ldots, n, \quad (4a)$$

$$v_j = v_i - 2(r_i P_i + x_i Q_i), (i, j) \in E \quad (4b)$$

From (4), we can derive that the voltages $\{v_k\}_{k=1,\ldots,N}$ and net power injections $\{s\}_{i=1,\ldots,N}$ satisfy the following equation:[2]

$$v_k = \sum_{i=1}^{n} R_{ki}(p_i^g - p_i^d) + \sum_{i=1}^{n} X_{ki}(q_i^g - q_i^d) + v_0,$$

where

$$R_{ki} := 2 \sum_{h \in \mathcal{P}_k \cap \mathcal{P}_i} r_h, \quad X_{ki} := 2 \sum_{h \in \mathcal{P}_k \cap \mathcal{P}_i} x_h.$$

Note that the voltages $\{v_i\}_{i\in N}$ are uniquely determined by the net power injections $\{s_i\}_{i=1,\ldots,n}$. As we assume that (q_i^g, q_i^d) are fixed for $i \in N$, we define $v_k^{nom} := \sum_{i=1}^{n} X_{ki}(q_i^g - q_i^d) + v_0$. Therefore, for $k = 1, \ldots, n$

$$v_k = \sum_{i=1}^{n} R_{ki}(p_i^g - p_i^d) + v_k^{nom}. \quad (5)$$

[1] In the optimization problem, we treat reactive power (q_i^g, q_i^d) as given constants rather than decision variables. The reason is for expository simplicity. The mechanisms developed in this paper can be extended to the case where reactive power consumption and generation are decision variables.

[2] The detailed derivation is given in [16].

In this paper, we will use the linear approximation model to illustrate how voltage constraints shape the market equilibrium and how we can design a market to let the externalities associated with voltage constraints internalized within private transactions.

III. An illustrative example

In this section, we present a simple case to highlight some important characteristics of the externalities associated with the voltage constraints and show how these externalities can be incorporated in a market mechanism. For illustrative purposes, we consider a simple distribution line with three nodes as shown in Figure 2 and we apply the linear approximation model as shown in Section II-C. We assume that node 0 is the feeder and nodes 1 and 2 are demand nodes with demand capacity of $\bar{p}_1^d = 9$ and $\bar{p}_2^d = 9$ and marginal utility (benefit) of $b_1 = 18$ and $b_2 = 20$ respectively. The power is supplied through the feeder node by an inverse supply function $\lambda(p^g) = 2p^g + 4$, where p^g is the electricity supply. We further assume that the two distribution lines, labeled $(0, 1)$ and $(1, 2)$, have identical electrical characteristics (i.e. line impedance) and that line losses are negligible.

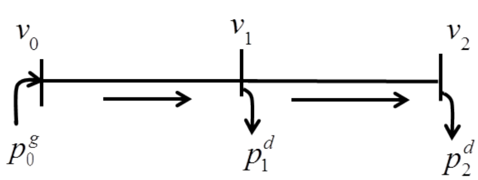

Fig. 2. Simple Distribution Network With 3 Nodes.

First, suppose that the voltage capacity constraints are ignored and generators and consumers are trading(selling/buying) electricity under a competitive market. In this case, the demand at the two demand nodes can be pooled into a single aggregate demand curve. As shown in Figure III, the optimal dispatch plan, determined by the intersection of the marginal utility function and the inverse demand curve, is that $(p_0^g, p_1^d, p_2^d) = (8, 0, 8)$, i.e. to inject 8 units power through node 0 and transfer it to node 2. The marginal cost of electricity at node 0 is 20 per unit.

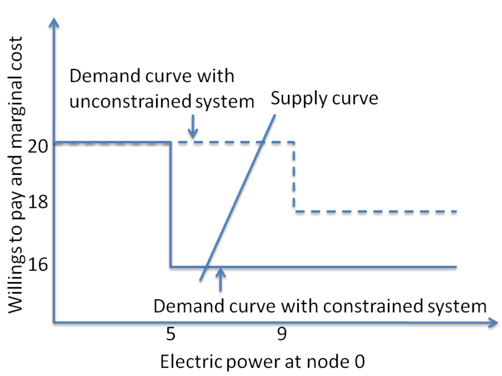

Fig. 3. Demand Supply curve at node 0.

However, if there is a voltage constraint on node 2, e.g. $v \geq \underline{v}$, this will bring an additional constraint on the power consumption on node 1 and 2, which can be written in the form of $p_1^d + 2p_2^d \leq \bar{p}$ according to equation (5). Suppose $\bar{p} = 10$. The previous dispatch plan will violate the voltage constraint. With the additional constraint, the utility maximization problem is to choose p_1^d and p_2^d so that the electricity supply p_0^g at node 0 can be dispatched at the maximum users' utility subject to the voltage constraints:

$$U(p^g) = \max_{p_1^d \leq \bar{p}_1^d, p_2^d \leq \bar{p}_2^d} b_1 p_1^d + b_2 p_2^d$$

subject to

$$p_1^d + p_2^d \leq p_0^g$$
$$p_1 + 2p_2 \leq \bar{p}$$

By solving the utility maximization problem parametrically for different values of p_0^g, we obtain a modified marginal utility function for power injection at node 0, as shown in Figure III. The inverse supply function and the modified marginal utility function now intersect at a price of 16 and an output level of 6, implying that the optimal dispatch is $(p_0^g, p_1^d, p_2^d) = (6, 2, 4)$. The shadow prices of electricity at nodes $0, 1, 2$ are $16, 18, 20$ respectively.

The above example illustrates that physical constraints (voltage capacity constraints) causes uncompensated use of parts of the network that lie outside of the private decisions. As a result, the private cost and the social cost diverge from each other in electricity transactions. Traditionally, these externalities associated with voltage capacity constraints are handled through the coordination by a central system operator. However, an alternative approach is to design new property rights and create markets for these rights so that the external effects associated with a transaction can be internalized in private decisions. For instance, we can introduce voltage capacity rights and establish the associated trading rules to define the quantity of voltage capacity rights that a trader needs to acquire in order to transfer power from one node to another node. In the rest of the paper, we will adopt this idea to design a decentralized electricity market for distribution networks to compensate the externalities associated with voltage capacity constraints as well as power losses.

IV. A Market Mechanism for voltage capacity constraints

For expository simplicity, we begin by assuming there are no distribution line losses and using the linear approximation model in (4). This will allows us to focus on the voltage capacity constraints in this section. In the next section, we will extend the idea to the nonlinear AC power flow model where the distribution line loss is considered. With the linear approximation model (4), the social welfare maximization problem is given as:

$$\max \quad \sum_{i \in N} B_i(p_i^d) - \sum_{i \in N} C_i(p_i^g) \tag{6}$$

$$\begin{array}{ll} \text{s.t.} & (2),(4) \\ \text{over} & p_i^g \in [\underline{p}_i^g, \bar{p}_i^g], p_i^d \in [\underline{p}_i^d, \bar{p}_i^d], i = 0, \ldots, n \\ & P_i, Q_i, \ell_i, v_i, i = 1, \ldots, n \end{array}$$

A. A market mechanism

In the following we will propose a decentralized market mechanism for the voltage capacity constraints so that the external effects on the voltage caused by a transaction can be internalized in private decisions. In other words, rather than letting a central regulator such as ISO to solve the social welfare maximization and determine the locational marginal prices, which implicitly reflect the voltage capacity constraints, we create a market which explicitly take into account the voltage capacity constraints so that any local decisions such as bilateral transactions between a pair of nodes will carry out their external effects on voltage.

The mechanism involves the definition of tradable voltage capacity rights and a trading rule that governs the exchange of these rights.

Definition 1 *A voltage capacity right entitles its owner to the right to send a unit of power through adjusting a bus's voltage. A fixed set of voltage capacity rights,* $\bar{\nu} := \{\bar{\nu}_k, k \in N \backslash \{0\}\}$, $\underline{\nu} := \{\underline{\nu}_k, i \in N \backslash \{0\}\}$, *are issued for each bus* k *and these rights are tradable.*[3]

The trading rule specifies the voltage capacity rights that traders must acquire in order to complete an electricity transaction. Electricity transactions usually specify the power transferred from one node to another without specifying the actually voltage change over the network. The following trading rule include the relationship between power transfers and voltage changes so that the voltage capacity rights can be enforced. Without loss of generality, we treat the bus 0 (feeder) as a base point and only need to define the trading rule that governs the transactions between the feeder and every other node in the network.

Definition 2 *The trading rule consists of a set of coefficients* $\beta = \{\bar{\beta}_i^k, \underline{\beta}_i^k | i, k \in N \backslash \{0\}\}$, *where the value* $\bar{\beta}_i^k$, $\underline{\beta}_i^k$ *represents the quantity of voltage capacity rights* $\bar{\nu}_k$, $\underline{\nu}_k$, *resp. on bus* k *that a trader needs to acquire in order to transfer a unit of power from node* i *to node* 0.

Using this trading rule, we can define terms of trade for bilateral transactions between an arbitrary pair of nodes. For instance, to transfer one unit of power from bus i to bus j requires $\bar{\beta}_{ij}^k \triangleq \bar{\beta}_i^k - \bar{\beta}_j^k$ units of voltage capacity rights $\bar{v}_k$ on bus k.

The market mechanism can thus be summarized as $(\bar{v}, \underline{v}, \beta)$. An efficient design of these terms is as follows: $\bar{\nu}_k$ is the voltage upper limit minus the nominal voltage at bus k and $\underline{\nu}_k$ is the nominal voltage minus the voltage lower limit at bus k, i.e.,

$$\bar{\nu}_k = \bar{v}_k - v_k^{nom}, \underline{\nu}_k = v_k^{nom} - \underline{v}_k; \tag{7}$$

[3]Notice that the voltage at the feeder 0 is fixed as customary.

and $\bar{\beta}_i^k$ ($\underline{\beta}_i^k$) is the (negative) voltage-loading factor: (negative) change on the voltage at bus k due to a unit power injection at node i. From equation (5), we know that an efficient design of β is

$$\bar{\beta}_i^k = R_{ki}, \underline{\beta}_i^k = -R_{ki}. \tag{8}$$

The following section will further explain why this market design is efficient.

B. Competitive Equilibrium and social optimum

Let us denote by $\bar{\xi}_i$ and $\underline{\xi}_i$ the prices of the voltage capacity right $\bar{v}_i$ and $\underline{v}_i$ respectively, and by λ_i the price of electricity at node $i \in N$. Denote by λ, p, and ξ the vectors of (λ_i), (p_i^g, p_i^d), and $(\bar{\xi}_i, \underline{\xi}_i,)$ for $i \in N$.

Definition 3 *A competitive equilibrium is a vector* (p, λ, ξ) *that satisfies the following three conditions:*

i) Each consumer maximize their net benefit and each generator maximize their net revenue given the price λ_i,

$$p_i^d \in arg \max_{p_i^d \in [\underline{p}_i^d, \bar{p}_i^d]} B_i(p_i^d) - \lambda_i p_i^d, \forall i \in N \tag{9}$$

$$p_i^g \in arg \max_{p_i^g \in [\underline{p}_i^g, \bar{p}_i^g]} \lambda_i p_i^g - C_i(p_i^g), \forall i \in N \tag{10}$$

ii) There should be no positive profit to be made by transfer of power from one node to another, i.e.,

$$\lambda_j = \lambda_i + \sum_{k=1}^n \bar{\xi}_k \beta_{ij}^k + \sum_k \underline{\xi}_k \underline{\beta}_{ij}^k, \forall i, j \in N. \tag{11}$$

iii) The price for voltage capacity rights is zero when there is excess supply:

$$\bar{\xi}_i [v_i - \bar{v}_i] = \underline{\xi}_i [v_i - \underline{v}_i] = 0, \forall i \in N. \tag{12}$$

Theorem 1 *Under the market mechanism* $(\mathbf{v}, \mathbf{v}, \beta)$*, a competitive equilibrium* (p, λ, ξ) *exists and is socially optimal.*

Proof: Firstly, by using equation (5), the social welfare problem is equivalent to the following optimization problem:

$$\begin{array}{ll} \max & \sum_{i \in N} B_i(p_i^d) - C_i(p_i^g) \\ \text{s.t.} & \sum_{i \in N} p_i^g - \sum_{i \in N} p_i^d = 0 \\ & (2, 5) \\ \text{over} & p_i^g \in [\underline{p}_i^g, \bar{p}_i^g], p_i^d \in [\underline{p}_i^d, \bar{p}_i^d], v_i \end{array}$$

For any p where $\underline{p}_i^g \leq p_i^g \leq \bar{p}_i^g$ and $\underline{p}_i^d \leq p_i^d \leq \bar{p}_i^d$, the Lagrangian function of this optimization problem is:

$$\begin{aligned} L &= -\sum_{i=0}^n B_i(p_i^d) + \sum_{i=0}^n C_i(p_i^g) \\ &\quad + \sum_{i=1}^n \omega_i \left(v_i - 2 \sum_{j=1}^n R_{ij}(p_j^g - p_j^d) - v_i^{nom} \right) \end{aligned}$$

$$+\lambda_0\left(-p_0^g+p_0^d-\sum_{j=1}^{n}(p_j^g-p_j^d)\right)$$
$$+\sum_{i=1}^{n}\bar{\xi}_i(v_i-\bar{v}_i)+\underline{\xi}_i(-v_i+\underline{v}_i)$$

Thus $(p, v, \omega, \lambda_0, \xi)$ is a primal-dual optimum if and only if, (p, v) is feasible, and

$$p_i^d \in \arg\max_{p_i^d\in[\underline{p}_i^d,\bar{p}_i^d]} B_i(p_i^d)-\lambda_i p_i^d, \forall i \in N, \tag{13}$$

$$p_i^g \in \arg\max_{p_i^g\in[\underline{p}_i^g,\bar{p}_i^g]} \lambda_i p_i^g - C_i(p_i^g), \forall i \in N, \tag{14}$$

where

$$\lambda_i = \sum_{k=1}^{n}\omega_k R_{ki} + \lambda_0, \tag{15}$$

and

$$\omega_i = -\bar{\xi}_i + \underline{\xi}_i, \tag{16}$$

$$\bar{\xi}_i[v_i-\bar{v}_i] = \underline{\xi}_i[\underline{v}_i - v_i] = 0, . \tag{17}$$

Therefore we know that by letting $\bar{\beta}_i^k = R_{ki}$, $\underline{\beta}_i^k = -R_{ki}$, (p, λ, ξ) is a competitive equilibrium.

On the other hand, given a competitive equilibrium (p, λ, ξ), it is easy to show that (13–17) hold. So to show the optimality of the competitive equilibrium, we only need to show that

$$\underline{v}_i \le v_i \le \bar{v}_i, \forall i = 1, \ldots, n. \tag{18}$$

Notice that the quantity of tradable voltage capacity rights demanded can not exceed the total quantity issued in (7). We have

$$\bar{v}_k - v_k^{nom} \ge \sum_i \bar{\beta}_i^k p_i = \sum_i R_{ki}p_i = v_k - v_k^{nom},$$
$$v_k^{nom} - \underline{v}_k \ge \sum_i \underline{\beta}_i^k p_i = -\sum_i R_{ki}p_i = v_k^{nom} - v_k.$$

Thus (18) holds and we show that the competitive equilibrium is social optimal. ■

Applying the preceding theorem, we have an interesting observation of the locational prices λ_i when all the branch buses $i = 1, \ldots, n$ are load buses. In this case, equation (5) tells that $v_i \le v_{\pi(i)} \le \ldots \le v_0$. Because in practice $\bar{v}_i > v_0$, we have $v_i < \bar{v}_i$. Therefore $\bar{\xi}_i = 0$ for all $i = 1, \ldots, n$. As a result, we have $\lambda_i > \lambda_{\pi(i)}$ for all $i = 1, \ldots, n$, meaning that the locational price increases along each path from the feeder 0 to the leaves.

V. The case with line losses

For the case with no losses, the distribution network exhibits the linearity property whereby the voltages are linear functions of power injections. It is therefore possible to decompose the voltage profile into power injections on each bus. Through a trading rule that embodies such a decomposition, a market could be established to determine voltage capacity prices and allocate the congestion rent efficiently.

In the presence of distribution line losses, things become more complicated, for each new electricity transfer could affect the distribution of line losses throughout the entire network. Line losses raise the problem of nonlinearity and even nonconvexity, adding significant complexities to the externality problem. There have been a series of work (see [17] and references therein) studying the convex relaxation of the social welfare maximization problem (3) where the quadratic equality constraint (1) is relaxed to the following inequality constraint:

$$\frac{P_{ij}^2+Q_{ij}^2}{v_i} \le \ell_{ij}.$$

Those studies establishes different sufficient conditions to guarantee the exact relaxation and it has been shown that most real radial distribution networks satisfy those sufficient conditions. Thus here we assume that the exact relaxation holds and focus on the convex relaxed problem which justify the use of the Lagrangian duality to develop price policies.

A. A market mechanism

To achieve economic efficiency when there are distribution line losses, it is desirable for the traders to pay for the marginal line losses. Thus a new type of economic rent is created for the line losses. The generalized market mechanism described below modifies the previous trading rule by specifying the voltage capacity rights and the compensation for power losses required for the completion of an electricity transaction. Since the distribution of the bus voltages and power losses depends on the actual power flow pattern, the trading rule is state dependent and varies continuously with time as the system condition evolves.

Formally, the trading rule consists two parts.

- The first part corresponds to the voltage capacity rights which is similar to those in the lossless case. It can be denoted as $\beta := \{\bar{\beta}_i^k, \underline{\beta}_i^k\}$ where $\bar{\beta}_i^k$ ($\underline{\beta}_i^k$, resp.) represents the quantity of voltage capacity rights $\bar{\nu}_k$ ($\underline{\nu}_k$, resp.) on bus k that a trader needs to acquire in order to inject a unit of real power at node i and deliver it to node 0. For an efficient mechanism, we can define $\bar{\nu}_k$ and $\underline{\nu}_k$ according to (7), and $\bar{\beta}_i^k$ and $\underline{\beta}_i^k$ as:

$$\bar{\beta}_i^k = \frac{(v_k - v_k^{nom})\,R_{ki}}{\sum_{j=1}^{n} R_{kj}p_j}; \tag{19}$$

$$\underline{\beta}_i^k = -\frac{(v_k - v_k^{nom})\,R_{ki}}{\sum_{j=1}^{n} R_{kj}p_j}. \tag{20}$$

By defining trading rule in this way, we have

$$\sum_{i=1}^{n}\bar{\beta}_i^k p_i = v_k - v_k^{nom}, \sum_{i=1}^{n}\underline{\beta}_i^k p_i = v_k^{nom} - v_k, \tag{21}$$

Roughly speaking, the coefficient β_i^k can be interpreted as the average loading factor of bus i for voltage on bus k.

- The second part corresponds to the economic rent for distribution line losses. We denote the trading rule as $\phi := \{\phi_i^k : i, k = 1, \ldots, n\}$ where ϕ_i^k represents the quantity of line losses on link $(k, \pi(k))$ that a trader needs to pay rents for in order to inject an unit of real power at node i and deliver it to node 0. We define the trading rule as:

$$\phi_i^k = \frac{L_k L_{k,i}}{\sum_{j=1}^n L_{k,j} p_j} \quad (22)$$

Here L_k denote the line losses on $(k, \pi(k))$ and $L_{k,i}$ is given as follows:

$$L_{k,i} = r_k \begin{cases} \frac{2P_k}{v_k} - \frac{P_k^2+Q_k^2}{v_k^2} R_{ki}, & \text{if } k \in \mathcal{P}_i, \\ -\frac{P_k^2+Q_k^2}{v_k^2} R_{ki}, & \text{otherwise.} \end{cases} \quad (23)$$

By defining trading rule in this way, we have

$$\sum_{i=1}^n \phi_i^k p_i = L_k. \quad (24)$$

Roughly speaking, ϕ_i^k can be interpreted as the average line losses from injecting power at node i.

Remark [Physical interpretation of $R_{k,i}, L_{k,i}$]: If we use the linear approximation model (4) to approximate the power flow and voltage, we have seen in section IV that $R_{k,i} = \frac{\partial v_k}{\partial p_i}$. Though the expression for $L_{k,i}$ is complicated, it has very nice interpretation if we use the linear approximation model (4) for power flow and voltage and use (1) to calculate the power loss L_k. Because the distribution line loss $L_k := r_k \ell_k = r_k \frac{P_k^2+Q_k^2}{v_k}$, we can calculate that $\frac{\partial L_k}{\partial p_i} = \frac{\partial L_k}{\partial P_k}\frac{\partial P_k}{\partial p_i} + \frac{\partial L_k}{\partial v_k}\frac{\partial v_k}{\partial p_i} = L_{k,i}$.

The trading terms: The trading terms for bilateral transactions between an arbitrary pair of nodes can be defined by combining multiple transactions. For instance, to transfer one unit of power from node i to node j requires $\bar{\beta}_{ij}^k := \bar{\beta}_i^k - \bar{\beta}_j^k$ and $\underline{\beta}_{ij}^k := \underline{\beta}_i^k - \underline{\beta}_j^k$ units of voltage capacity rights $\bar{v}_k$ and $\underline{v}_k$ respectively on bus k, and economic rents for $\phi_{ij}^k := \phi_i^k - \phi_j^k$ units of line losses on link $(k, \pi(k))$. The market mechanism can thus be summarized as $(\bar{v}, \underline{v}, \beta, \phi)$. The distribution charge can be decomposed into two components: a rent for voltage congestion and a rent for the externalities associated with distribution line losses. The rent for a bus voltage binding is zero when it is not congested and it becomes positive only when the voltage reaches its capacity, whereas the rent for line losses is nonzero most of the time, regardless of the capacity condition.

B. Competitive equilibrium and social optimum

Let us denote by η_k the economic rent for the line losses on line $(i, \pi(i))$. Denote by η the vector of (η_i) for $i = 1, \ldots, n$. The definition of competitive equilibrium (p, λ, ξ, η) remains virtually the same as in Definition 3 with the exception that condition ii) needs to be replaced by the following version:

ii). There should be no positive profit to be made by transfer of power from one node to another. Specifically, the transaction of delivering a unit of power from node i to j requires the payment of 1) the electricity purchase price p_i 2) the voltage binding cost $\bar{\xi}_k \bar{\beta}_{ij}^k + \underline{\xi}_k \underline{\beta}_{ij}^k$ for all $k = 1, \ldots, n$ 3) the cost for line losses, $\eta_k \phi_{ij}^k$. At competitive equilibrium, the net profit for such a transaction should be zero, i.e.,

$$\lambda_j = \lambda_i + \sum_{k=1}^n (\eta_k \phi_{ij}^k + \bar{\xi}_k \bar{\beta}_{ij}^k + \underline{\xi}_k \underline{\beta}_{ij}^k). \quad (25)$$

Theorem 2 *Under the market mechanism $(\bar{v}, \underline{v}, \beta, \phi)$, there exists such a competitive equilibrium (p, λ, ξ, μ) where p is socially optimal in the presence of line losses.*

Proof: For any p where $\underline{p}_i^g \le p_i^g \le \bar{p}_i^g$ and $\underline{p}_i^d \le p_i^d \le \bar{p}_i^d$, the Lagrangian is given by:

$$\begin{aligned} L = & -\sum_{i=0}^n B_i(p_i^c) + \sum_{i=0}^n C_i(p_i^g) + \sum_{i=1}^n r_i \ell_i \\ & + \sum_{i=0}^n \lambda_i (P_i - \sum_{j \in \delta(i)} P_j - \sum_{j \in \delta(i)} r_i \ell_i - p_i^g + p_i^d) \\ & + \sum_{i=0}^n \theta_i (Q_i - \sum_{j \in \delta(i)} Q_j - \sum_{j \in \delta(i)} x_j \ell_j - q_i^g + q_i^d) \\ & + \sum_{i=1}^n w_i (v_i - v_{\pi(i)} - 2(r_i P_i + x_i Q_i) + (r_i^2 + x_i^2)\ell_i) \\ & + \sum_{i=1}^n \mu_i (\frac{P_i^2 + Q_i^2}{v_i} - \ell_i) \\ & + \sum_{i=1}^n \bar{\gamma}_i (v_i - \bar{v}_i) + \underline{\gamma}_i (-v_i + \underline{v}_i) \end{aligned}$$

Thus $(p, P, Q, \ell, v, \lambda, \theta, \omega, \mu, \gamma)$ is a primal-dual optimum, if and only if, (p, P, Q, ℓ, v) are feasible, and,

$$p_i^d \in \arg\max_{p_i^d \in [\underline{p}_i^d, \bar{p}_i^d]} B_i(p_i^d) - \lambda_i p_i^d, \forall i \in N$$

$$p_i^g \in \arg\max_{p_i^g \in [\underline{p}_i^g, \bar{p}_i^g]} \lambda_i p_i^g - C_i(p_i^g), \forall i \in N$$

and for $i = 1, \ldots, n$,

$$\frac{\partial L}{\partial P_i} = -\lambda_{\pi(i)} + \lambda_i + 2\mu_i \frac{P_i}{v_i} - 2w_i r_i = 0, \quad (26)$$

$$\frac{\partial L}{\partial Q_i} = -\theta_{\pi(i)} + \theta_i + 2\mu_i \frac{Q_i}{v_i} - 2w_i x_i = 0, \quad (27)$$

$$\frac{\partial L}{\ell_i} = r_i - \lambda_{\pi(i)} r_i - \theta_{\pi(i)} x_i - \mu_i + w_i (r_i^2 + x_i^2) = 0 \quad (28)$$

$$\frac{\partial L}{v_i} = w_i - \mu_i \frac{P_i^2 + Q_i^2}{v_i^2} - \sum_{k \in \delta(i)} w_k + \gamma_i = 0, \quad (29)$$

where $\gamma_i := \bar{\gamma}_i - \underline{\gamma}_i$ and

$$\bar{\gamma}_i [v_i - \bar{v}_i] = \underline{\gamma}_i [-v_i + \underline{v}_i] = 0.$$

For any $i = 1, \ldots, n$, summing up equation (29) for all $j \in \Delta(i)$, we have,

$$w_i = \sum_{j \in \Delta(i)} \left(-\gamma_i + \mu_i \frac{P_i^2 + Q_i^2}{v_i^2} \right) \quad (30)$$

Summing up equation (26) along the path from bus 0 to bus i, we have,

$$\lambda_0 = \lambda_i + \sum_{j \in \mathcal{P}_i} \left(\mu_j \frac{2P_j}{v_j} - 2r_j w_j \right). \tag{31}$$

Substituting (30) into (31), we have

$$\begin{aligned} \lambda_0 &= \lambda_i + \sum_{j \in \mathcal{P}_i} \mu_j \frac{2P_j}{v_j} \qquad (32) \\ &\quad - \sum_{j \in \mathcal{P}_i} 2r_j \left(\sum_{k \in \Delta(j)} \left(-\gamma_k + \mu_k \frac{P_k^2 + Q_k^2}{v_k^2} \right) \right) \\ &= \lambda_i + \sum_{k \in \mathcal{P}_i} \mu_k \frac{2P_k}{v_k} + \sum_{k=1}^{n} R_{ki} \left(\gamma_k - \mu_k \frac{P_k^2 + Q_k^2}{v_k^2} \right) \\ &= \lambda_i + \sum_{k=1}^{n} \frac{\mu_k}{r_k} L_{k,i} + \sum_{k=1}^{n} R_{ki} \bar{\gamma}_k - \sum_{k=1}^{n} 2R_{ki} \underline{\gamma}_k \end{aligned}$$

Defining β, ϕ according to (19, 23), and let

$$\bar{\xi}_k = \begin{cases} \frac{\sum_{j=1}^{n} R_{kj} p_j}{v_k - v_k^{nom}} \bar{\gamma}_k, & \text{if } v_k = \bar{v}_k; \\ \bar{\gamma}_k = 0, & \text{otherwise}, \end{cases} \tag{33}$$

$$\underline{\xi}_k = \begin{cases} \frac{\sum_{j=1}^{n} R_{kj} p_j}{v_k - v_k^{nom}} \underline{\gamma}_k, & \text{if } v_k = \underline{v}_k; \\ \underline{\gamma}_k = 0, & \text{otherwise}, \end{cases} \tag{34}$$

$$\eta_k = \frac{\sum_{j=1}^{n} L_{k,j} p_j}{L_k} \frac{\mu_k}{r_k}. \tag{35}$$

Then we have for any (i, j),

$$\lambda_j = \lambda_i + \sum_{k=1}^{n} (\eta_k \phi_{ij}^k + \bar{\xi}_k \bar{\beta}_{ij}^k + \underline{\xi}_k \underline{\beta}_{ij}^k),$$

which means that (p, λ, ξ, η) is a competitive equilibrium.

∎

VI. A Numerical Example

To illustrate the results, we consider a three node distribution line (as shown in Figure 2) with two identical distribution lines, each of which is characterized by an inductance of $x = 3.88$ ohms and a resistance of $r = 1.6$ ohms. The nominate voltage value is assumed to be $V_0 = 12.35$ kV. The voltage capacity constraints at node 1 and 2 is $[0.95V_0, 1.05V_0]$. The cost function assumed for each generator is in the form of $1/2ap^2 + bp$ and the utility function assumed for each consumer is in the form of $-1/2cp^2 + dp$. Table VI displays the value of the parameters for generation and consumption.[4] For expository simplicity, the reactive power injection at each node is assumed to be zero.

Under the above assumptions, the market mechanism is characterized by the issuance of $\bar{v}_k = (1.05^2 - 1)V_0^2$ and $\underline{v}_k = (1 - 0.95^2)V_0^2$ of voltage capacity rights for each node $i = 1, 2$ and a trading rule (which is continuously updated in the market trading process). As the market reaches a competitive equilibrium, the trading rule is characterized by (β, ϕ) as displayed in Table VI. The resulting competitive equilibrium is summarized in table VI.

[4]Note that in order to amplify the effect of voltage capacity constraints, we scale up the value of resistance, reactance, consumption and generation compared to a normal distribution feeder.

TABLE I

PARAMETERS FOR GENERATION AND CONSUMPTION

	a	b	c	d	$[\underline{p}^g, \bar{p}^g]$	$[\underline{p}^d, \bar{p}^d]$
node 0	0.1	0.3	0.2	1.2	[0.45, 4.56]	[0.36, 0.66]
node 1	0.2	0.1	0.2	1.5	[0.06, 0.21]	[1.02, 2.52]
node 2	0.2	0.05	0.1	2.2	[0.09, 0.50]	[1.65, 2.88]

TABLE II

TRADING RULES

k	$\bar{\beta}_1^k$	$\bar{\beta}_2^k$	$\underline{\beta}_1^k$	$\underline{\beta}_2^k$	ϕ_1^k	ϕ_2^k
1	3.8377	3.8377	-3.8377	-3.8377	-0.0288	-0.0288
2	3.6625	7.3250	-3.6625	-7.3250	-0.0003	-0.0180

TABLE III

COMPETITIVE EQUILIBRIUM

k	p_k^d	p_k^g	v_k	λ_k	$\underline{\xi}_k$	$\bar{\xi}_k$	η_k
0	0.660	3.236	152.5225	0.6236			
1	1.020	0.210	143.1770	1.3432	0	0	9.1125
2	2.105	0.480	137.6516	1.9895	0.1238	0	11.5616

We can check the competitive equilibrium condition (25) by using the values provided by Table VI and Table VI. As examples, for node 0 and 1, we have $\lambda_0 = \lambda_1 + \underline{\beta}_1^2 \underline{\xi}_2 + \phi_1^1 \eta_1 + \phi_1^2 \eta_2$; for node 0 and 2, we have $\lambda_0 = \lambda_2 + \underline{\beta}_2^2 \underline{\xi}_2 + \phi_2^1 \eta_1 + \phi_2^2 \eta_2$. Note that the convex relaxation is checked to be exact for this distribution line.

VII. Discussion and Conclusion

In this paper, we present a market mechanism for a radial distribution network which internalize those externalities within private decisions such as local trading transactions, by defining trading rules which efficiently allocate the externalities to individuals in the network. Specifically, we focus on the external costs associated with voltage constraints and distribution line losses. We show that a competitive market could be established for distribution services and electricity to achieve a social optimum. This paper serves as an initial step to provide guidelines for creating a decentralized electricity market on distribution networks.

As we have seen, the market mechanism and the analysis with distribution line losses are more more complicated than the lossless case. We would like to further discuss the limitation and related future work to complement our design and analysis.

Firstly, we note that in order to ensure (21) and (24), we define a complicate trading rule for β and ϕ as in (19,23) which requires the global system information. Conditions of (21) and (24) mean that the total quantities of voltage capacity rights and line losses requested by buses should equal the true voltage capacity rights and line losses used in the network. These conditions are similar to the budge balance conditions required in other resource allocation problems [18]. If the trading rule does not need to satisfy such budget balance conditions, we can define simpler trading rules which requires less information but still guarantee the efficiency of the market mechanism. Future work involves exploring different trading rules to suit different requirements and information constraints.

Secondly, compared with Theorem 1, the statement in Theorem 2 is weaker. Theorem 2 guarantees the existence of such a competitive equilibrium with social optimal welfare but does not guarantee the optimality for every competitive equilibrium. This is because of the nonlinear power flow equations in (1). Given a power injection profile (p, q), (1) will determine the power flow and voltage over the network but the solutions are generally non-unique, and there are cases where a solution does not exist [19]. Because of the nonlinearity of the AC power flow, more conditions are required to satisfy for a competitive equilibrium to be socially optimal. Presumably, we can define the competitive equilibrium in a more complicated way, e.g. including additional conditions on P, Q, v and additional price terms, to make the competitive equilibrium and the social optimum are equivalent. However, this additional complexity may affect the applicability of the market mechanism.

Thirdly, we have focused on the market design and the competitive equilibrium but have not specified any trading process to reach the equilibrium as done in [10]. One way to design the trading process is to use the dual gradient algorithm to solve the social welfare maximization problem. But there are other possibilities as well, e.g., a trading process similar to the one in [10]. One future direction is to develop different trading processes, study their convergences, and explore their economic insights.

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