Time-varying utility functions

Authors

***Abstract*—A key challenge faced by service operators (Load Servicing Entities LSEs) in power networks today is optimally allocating limited electrical power to users with private information, such as consumption schedule and utility functions, in a decentralized manner. Much of the existing power distribution infrastructure today also limits non-power information flow, primarily enabling the communication of real-time pricing from the LSEs to the users via a one-way communication protocol, despite users' changing needs and utilities. This paper studies the properties of a distributed and iterative electrical power allocation algorithm for a decentralized dynamic power allocation problem comprising N users and R LSEs, where the users' utility and LSEs' cost functions changes at the same time scale as each iteration of the algorithm (either due to users' changing needs or real-time pricing signal from the suppliers). We approach this using the dual-gradient method. We assume users have controllers (or battery-powered back up sources) Lina, this is in line with the value of storage idea you talked about. We'll expand on it. , that ensures the effect of changes in their power consumption and utility functions will be smooth between successive iterations and show how the market can be regulated via the price of power. Further, we derive a bound on how well our distributed algorithm with one-way communication tracks the optimal price at each time-step.**

I. Introduction

Of vital importance in the energy management infrastructure today are fast, efficient and scalable distributed algorithms that take into account consumers' constantly changing power needs. Increasingly, network sensing and computation have been studied on power networks with several distributed algorithms proposed [1–5].

While several distributed algorithms have been proposed for power networks, many of them assume bi-directional communication in coordinating computation between consumers and LSEs. In addition, not much attention has been paid to the dynamic power allocation problems that take into account users' constantly changing behaviors, as well as the dynamic cost of power generation. These challenges motivate the problems addressed in this paper – developing and investigating a dynamic distributed energy management using one-way communication, where the respective utility and power generation functions of the users and LSEs change at the time same time scale as the iterative algorithm.

We, in addition, assume that that the utilities users' get from consumption change at each time-step of the algorithm. Based on the assumption that users have controllers that prevent drastic change in the magnitude of their utility functions; and that the capacity of LSEs between successive time-steps is bounded, we show:

- how to appropriately regulate the market via bounded changes in power pricing at each time-step, and
- how to track the optimal solution (price) by our distributed algorithm, via a one-way communication protocol.

A. Literature Review

B. Contributions of This Work

The rest of the paper is organized as follows: Following notation and definitions, we introduce the system model, underlying assumptions and distributed solution in Section II. Sections III and IV respectively present a convergence analysis and numerical illustration of our algorithm. We make our conclusions in Section V.

Distributed solutions to power allocation problems are not new, particularly using common first-order methods like dual-descent, gradient methods. Optimization problems with time-changing objective functions or constraint set arise in several contexts – for instance, scheduling and tracking. Dynamic optimization problems are typically solved as a sequence of static problems. Typically separate time-scales are assumed so that the algorithm converges in between changes in the algorithm. In dynamic optimization problems, results on convergence depend on the rate at which the algorithm converges relative to how fast the problem is changing. In summary, if the objective functions $U_i(t)$ changes sufficiently slowly, static algorithms can fairly easily be adapted for dynamic problems. That assumption on time-scales is not satisfactory because real-time pricing from LSEs and constantly changing consumers' needs affect their utility at each time-step. Hence, we consider a system where the users utility functions changes at the same time-scale as each iteration of the algorithm.

II. Model and Algorithm

We consider a power distribution system comprising N users and R Load Servicing Entities (LSEs), whose objective at time t is to solve a distributed dynamic power allocation problem. We abstract away specific power network constraints

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for clarity in presentation, and assume that each user has a dynamic utility function. We consider the problem of solving in a distributed manner, the social welfare problem of allocating power to the users formally formulated as the following optimization problem:

$$\begin{aligned} \underset{\mathbf{q},\ \mathbf{s}}{\text{maximize}} \quad & \sum_{i=1}^{N} U_i^t(\mathbf{q}_i(t)) - \sum_{r=1}^{R} C_r^t(s_r(t)) \\ \text{subject to} \quad & \mathbf{q}_i(t) \in \mathcal{Q}_i \text{ for } i = 1, \dots, N, \\ & s_r(t) \in \mathcal{S}_r, \text{ for } r = 1, \dots, R, \\ & \sum_{i=1}^{N} \mathbf{q}_i(t) \leq \mathbf{s}(t), \end{aligned} \tag{1}$$

where $U_i^t(\cdot)$ is user i's utility function at time t, dependent on the amount of power $\mathbf{q}_i(t) \in \mathbb{R}^R$ allocated at time t; and $C_r^t(\cdot)$ is the cost of producing $s_r \in \mathbb{R}$ units of power at time t. Let $\mathbf{q} = [\mathbf{q}_1, \dots, \mathbf{q}_N] \in \mathbb{R}^{R \times N}$ and $\mathbf{s} = (s_1, \dots, s_R) \in \mathbb{R}^R$. The rth element of $\mathbf{q}_i(t)$ represents the amount of power supplied by the rth LSE. The static case of (1) comprising a single LSE and N users was studied in [6]. Our objective is to present a distributed solution to the dynamic optimization problem in (1) through a one-way communication (coordination) protocol that achieves

- a proportionally fair allocation,
- an efficient allocation policy,
- alongside a convergence analysis with bounds and guarantees on how the optimal solution changes.

A proportionally fair allocation policy can be achieved by an appropriate choice of utility functions by each user [7]. If the consumption-generation inequality is binding, efficiency in the allocation policy is achieved (since power generated at each time-step is used up). The assumed one-way communication protocol in coordinating between the users and LSEs is enabled by the following operations:

Operation 1 (One-way Communication): At each time-step, the Load Servicing Entities broadcasts a scalar message (referred to as the unit cost of power), to the users.

Operation 2 (Feedback Information) At each time-step, the Load Servicing Entities measure the difference between their total power capacity and their supply; that is, $\mathbf{q}(t)\mathbf{1} - \mathbf{s}(t)$

We assume the cost of producing power as well as the users' utility functions are dynamic at each time step. This could be either because of the changing needs of the user at each time or due to the real-time price signals received from the array of suppliers the user has access to. Further, we make further assumptions on the utility functions.

Assumption 1. *We assume that each user i has an upper bound on how much its utility function changes between consecutive time steps; that is,*

$$\|U_i^{t+1}(\mathbf{q}_i) - U_i^t(\mathbf{q}_i)\| \leq \alpha. \tag{2}$$

Assumption 2. *(Distributed and Private): We assume that for each user i and LSE r, their respective utility and cost functions $U_i(\cdot)$ and $C_r(\cdot)$ and decision variables $\mathbf{q}_i$ and s_r are private; that is, they are known only by the user or LSE.*

Definition 1. *A differentiable function $U_i^t(\cdot)$ is strongly convex if for all users i and time t, given any two vectors $\mathbf{q}_1$ and $\mathbf{q}_2$, we have $(\mathbf{q}_1 - \mathbf{q}_2)^T[\nabla U_i^t(\mathbf{q}_1) - \nabla U_i^t(\mathbf{q}_2)] \geq \mu_i^t \|\mathbf{q}_1 - \mathbf{q}_2\|^2$, where $\mu_i^t > 0$ is the strong concavity constant for user i's utility function $U_i^t(\mathbf{q}_i)$ at time t.*

Definition 2. *A function $U(\mathbf{q})$ is strongly concave if $-U(\mathbf{q})$ is strongly convex.*

Assumption 3. *(Strong Concavity): We assume the users utility functions $U_i^t(\mathbf{q})$ are strongly concave with parameter μ_i^t.*

Assumption 4. *(Lipschitz Gradients): The gradients of the utility function of each user i is Lipschitz continuous at each time-step; that is, for all vector pairs $\mathbf{q}_1$ and $\mathbf{q}_2$, the following $\|\nabla U_i^t(\mathbf{q}) - \nabla U_i^t(\mathbf{q}_2)\| \leq L_i^t \|\mathbf{q}_1 - \mathbf{q}_2\|$, where $L_i^t < \infty$ is the Lipschitz constant.*

Assumptions 3 and 4 above imply that the (local) objective function in (1) is strongly concave with Lipschitz gradients. We will assume that for all time t, $\mu_i = \max_t\{\mu_i^t\}$ is the global concavity constant, and $L_i = \min_t\{L_i^t\}$ is the global Lipschitz constant. These assumptions are typical in the convergence analysis of descent algorithms. Distributed algorithms that solve (1) using Operations 1 and 2 can be achieved via duality theory. Then the dual of (1) is:

$$\begin{aligned} \underset{\mathbf{p}}{\text{minimize}} \quad & D(\mathbf{p}) \\ \text{subject to} \quad & \mathbf{p} \geq 0, \end{aligned} \tag{3}$$

where $D(\cdot)$ and $\mathbf{p} \in \mathbb{R}^R$ are respectively the dual function and dual variable; and $D(\cdot)$ is:

$$\begin{aligned} D(\mathbf{p}) &= \underset{\mathbf{q},\ \mathbf{s}}{\text{maximize}} \sum_{i=1}^{N} U_i^t(\mathbf{q}_i) - \sum_{r=1}^{R} C_r^t(s_r) - \mathbf{p}\left(\sum_{i=1}^{N} \mathbf{q}_i - \mathbf{s}\right) \\ &= \sum_{i=1}^{N} U_i(\mathbf{q}_i(\mathbf{p})) - \sum_{r=1}^{R} C_r^t(s_r(\mathbf{p})) - \mathbf{p}\left(\sum_{i=1}^{N} \mathbf{q}_i(\mathbf{p}) - \mathbf{s}\right). \end{aligned} \tag{4}$$

The respective local problems for each user i and LSE r are to solve

$$\mathbf{q}_i(\mathbf{p}) = \arg\max_{\mathbf{q}_i \in \mathcal{Q}_i} U_i(\mathbf{q}_i) - \mathbf{p}^T\mathbf{q}_i, \tag{5}$$

$$s_r(p_r) = \arg\max_{s_r \in \mathcal{S}_r} -C_r(s_r) - p_r s_r. \tag{6}$$

The structure of the problem enables us claim the following result:

Lemma 1. *(Strong Duality): Consider* (1)*, and suppose Assumptions 3 and 4 hold, and let $\mathbf{p}^*(t)$ be the optimal solution to to* (3)*, then $\mathbf{q}(\mathbf{p}^*) = \{\mathbf{q}_i(\mathbf{p})\}_{i=1}^N$ (cf.* (5)*) is the optimal solution to* (1).

Proof: Convexity of the problem coupled with the constraints $\sum_{i=1}^N \mathbf{q}_i \leq \mathbf{s}$, and $\mathbf{q}_i \in \mathcal{Q}_i$ ensures that (1) satisfies Slater's condition, yielding a zero duality gap [8, Chapter 5]. ■

Proposition 1. *Consider the problem* (1) *and suppose Assumptions 3 and 4 hold, then the dual function* (4) *is strongly convex with parameter $\hat{\alpha}$ – place holder.*

Proof: Working out the details. ■

Algorithm 1 A distributed dual-descent Algorithm for locally computing optimal generation and power allocations.

Initialization: Suppliers set initial price $\mathbf{p}(0)$;
for $t = 0, \ldots$ **do**
  Suppliers broadcast $\mathbf{p}(t)$
  for Users $i = 1, \ldots, N$ **do**
    User i receives $\mathbf{p}(t)$ and solves (5)
  end for
  Suppliers measure $\mathbf{q}(t)\mathbf{1} - \mathbf{s}(t)$
  and compute next price
  $\mathbf{p}(t+1) = \mathbf{p}(t) - \gamma(\sum_{i=1}^{N} \mathbf{q}_i(t) - \mathbf{s})$
end for

In [6] we showed how to appropriately select parameters to achieve a linear convergence rate in the solution of (1) when the parameters and problem variables are not time-varying. Next, we study convergence properties of Algorithm 1.

III. Convergence Analysis of Algorithm 1

In this section, we derive bounds on how the optimal changes between successive iterations and how far the price computed in Algorithm 1 is from the optimal price at each iteration of the algorithm.

Definition 3. *We say a function $U_i^t(\cdot)$, has Lipschitz continuous gradients in the variable t if*

$$\|\nabla U_i^t(q) - \nabla U_i^{t+1}(q)\| \leq \alpha.$$

To derive a bound on the difference between the optimal dual variable between successive iterations,

$$\|p^*(t+1) - p^*(t)\| \leq \kappa, \tag{7}$$

for some $\kappa > 0$, we will first derive similar bounds for the primal variables and exploit convexity of (1) and Lemma 1.

Proposition 2. *Consider the system problem* (1)*, coupled with Assumptions 3 and 4, the optimal primal variable, between consecutive iterates, for each user is bounded by*

$$\|\mathbf{q}_i^*(t+1) - \mathbf{q}_i^*(t)\| \leq \frac{\alpha}{\mu}.$$

Proof: Staring with the definition on strong concavity, we have that $-[\nabla U_i^t(\mathbf{q}_1) - \nabla U_i^t(\mathbf{q}_2)]^T(\mathbf{q}_1 - \mathbf{q}_2) \geq \mu_i^t\|\mathbf{q}_1 - \mathbf{q}_2\|^2$. If we expand the definition, it follows that

$$\nabla U^t(\mathbf{q}(t+1))^T(\mathbf{q}^*(t) - \mathbf{q}^*(t+1)) \geq \mu\|\mathbf{q}^*(t) - \mathbf{q}^*(t+1)\|^2$$
$$-\nabla U^{t+1}(\mathbf{q}^*(t))^T(\mathbf{q}^*(t) - \mathbf{q}^*(t+1)) \geq \mu\|\mathbf{q}^*(t) - \mathbf{q}^*(t+1)\|^2.$$

Summing them up, yields Lina, there's a wrong assumption here that needs to be corrected. It's essentially a substitution of the optimal dual variable at certain places.

$$2\mu\|q^*(t) - q^*(t+1)\|^2 \leq \langle\nabla U^t(q^*(t+1)) - \nabla U^{t+1}(q^*(t)), q^*(t) - q^*(t+1)\rangle. \tag{8}$$

Recall that $\|U^t(q^*(t+1))\| \leq \alpha$ and $\|U^{t+1}(q^*(t))\| \leq \alpha$. The Cauchy-Schwarz inequality applied to (8) implies that

$$2\mu\|q^*(t) - q^*(t+1)\|^2 \leq \langle\nabla U^t(q^*(t+1)) - \nabla U^{t+1}(q^*(t)), q^*(t) - q^*(t+1)\rangle \leq \|\nabla U^t(q^*(t+1)) - \nabla U^{t+1}(q^*(t))\|\|q^*(t) - q^*(t+1)\|. \tag{9}$$

From (9),

$$2\mu\|q^*(t) - q^*(t+1)\| \leq \|\nabla U^t(q^*(t+1)) - \nabla U^{t+1}(q^*(t))\|,$$

and applying the triangle inequality on the RHS of the preceding inequality means that

$$2\mu\|q^*(t) - q^*(t+1)\| \leq \|\nabla U^t(q^*(t+1)) - \nabla U^{t+1}(q^*(t+1))\| + \|\nabla U^{t+1}(q^*(t+1)) - \nabla U^t(q^*(t))\| + \|\nabla U^t(q^*(t)) - \nabla U^{t+1}(q^*(t))\| \leq 2\alpha.$$

Thence, from the inequalities above, we conclude that

$$\|q^*(t) - q^*(t+1)\| \leq \frac{\alpha}{\mu}.$$

■

Proposition 3. *The optimal dual variable between consecutive iterations satisfies $\|p^*(t+1) - p^*(t)\| \leq \kappa$, for some $\kappa < \infty$.*

Proof: We can easily show that Proposition 3 by using the strong duality result from Lemma 1. assuming a bounded capacity between consecutive iterations $\|Q(t) - Q(t+1)\| \leq \gamma$ or by using Lemma 1 (strong duality) and the fact that $\|\mathbf{q}^*(t) - \mathbf{q}^*(t+1)\| \leq \alpha/\mu$. ■

In addition to using the strong duality result of Lemma 1, it is also possible to show that Proposition 3 holds by a slight adjustment to the problem model where a bound on the change in capacity $Q_i(t)$ between consecutive iterations is assumed. In particular, if the feasible set of $\mathbf{q}_i(t)$ is $Q_i(t)$, then Proposition 3 holds if $\|Q(t) - Q(t+1)\| \leq \gamma,\ \gamma \leq \infty$.

Given the bound on the system price (regulation parameter), of interest is how well Algorithm 1 tracks the optimal price at each iteration of the algorithm. Here, we present a result showing that the price at iteration of Algorithm 1 falls within an ε-ball of the optimal price.

Theorem 1. *(Main Result): Consider the system problem* (1) *and its dual problem* (3)*. The distance between the optimal price $\mathbf{p}^*(t+1)$ and the price generated Algorithm 1 at each iteration t is*

$$\|\mathbf{p}(t+1) - \mathbf{p}^*(t+1)\| \leq \frac{\kappa}{1-c} + c^t\left(\|\mathbf{p}(0) - \mathbf{p}(t)\| - \frac{\kappa}{1-c}\right), \tag{10}$$

where κ comes from Proposition 3 and c is a constant.

Proof: The first term on the RHS of the inequality a product of $f_t(\mathbf{p}(t))$, and f_t is a $c-$contractive parameter.

$$
\begin{aligned}
\|\mathbf{p}(t+1)-\mathbf{p}^*(t+1)\| &\leq \|\mathbf{p}(t+1)-\mathbf{p}^*(t)\| \\
&\quad + \underbrace{\|\mathbf{p}^*(t)-\mathbf{p}^*(t+1)\|}_{\leq \kappa} \\
&\leq c\|\mathbf{p}(t)-\mathbf{p}^*(t)\| + \kappa \\
&\leq c^t\|\mathbf{p}(0)-\mathbf{p}^*(0)\| + \sum_{i=0}^{t} c^i \kappa \\
&= c^t\|\mathbf{p}(0)-\mathbf{p}^*(0)\| + \frac{1-c^t}{1-c}\kappa \\
&= \frac{\kappa}{1-c} + c^t\left(\|\mathbf{p}(0)-\mathbf{p}(t)\| - \frac{\kappa}{1-c}\right).
\end{aligned}
$$

■

We have differentiable utility functions, with Lipschitz continous gradients. Strong concavity of the objective functions imply that the solutions $\mathbf{s}^*(\mathbf{p})$, $\mathbf{q}^*(\mathbf{p})$ to (1) exist and are unique. If we take the derivative of the Lagrangian with respect to the decision variables at time t, we have

$$
-C'(\mathbf{s}_r^*(t), t) + \mathbf{p}(t) = 0, \tag{11}
$$

and

$$
U'(\mathbf{q}_i^*(t), t) - \mathbf{p}(t) = 0, \tag{12}
$$

The dual function is differentiable with derivative is

$$
\mathbf{s}^*(\mathbf{p}(t)) - \sum_{i=1}^{N} \mathbf{q}^*(\mathbf{p}(t)).
$$

We will exploit strong concavity of the utility functions and Lipschitz continuity of their gradients to get a strongly convex dual function with some global convexity parameter.

IV. Numerical Illustrations

In this section are illustrations of the results in the preceding sections with problem parameters .

V. Conclusions

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