Achieving Real-time Economic Dispatch in Power Networks via a Saddle Point Design Approach

Xuan Zhang, Na Li and Antonis Papachristodoulou

***Abstract*—The increasing distributed energy resources introduce rapid and fast fluctuations in power supply and demand. As a result, the control and economic optimization for power networks need to operate at faster time-scales for reliability and economical efficiency. In this paper, we consider the problem of achieving real-time economic dispatch for power networks with controllable loads. We first present the dynamics over the transmission network under exogenous disturbances and formulate an economic dispatch problem with the power network congestion constraints. We then propose a saddle point design framework in which system dynamics and the optimization problem are combined to derive a completely distributed controller. As a result, the closed-loop system asymptotically converges to an equilibrium point where the optimization problem is solved. An example using the IEEE 14-bus network illustrates the effective performance of the control scheme.**

***Index Terms*—Power system, optimization, distributed control, frequency control, reverse-engineering.**

I. Introduction

Conventional frequency control in power networks is designed and implemented in a layered way, including primary, secondary and tertiary control [1]. When disturbances occur, the primary frequency control operates at a time-scale of seconds to adjust the mechanical power input of generators based on local frequency deviations. Then secondary frequency control, aka Automatic Generation Control (AGC), takes place at a time-scale of 10 seconds to minutes to restore nominal frequency and scheduled interchanges of tie-lines. Lastly, the tertiary control, aka Economic Dispatch (ED), operates at a time-scale of minutes to determine the economically-efficient nominal operating point of the mechanical power generation and power interchanges of tie-lines.

This layered frequency control mechanism works well because of high predictability and low uncertainty in the system [2]. However, as renewable energy and dynamic demand are introduced, the power grid will face increasing and large uncertainties and variability in both supply and demand. In order to maintain system-wide efficiency, reliability and robustness under these changes, recent work has focused on the redesign of conventional frequency control approaches that combines different layers/time-scales together, in either distributed, decentralized or centralized manner [2]-[11]. For example, [2], [3], [4] presented distributed real-time control architectures which merged primary, secondary, and tertiary frequency control for power networks with controllable loads, so as to achieve system-wide efficiency and reliability. In [5], [6], [7], [8], [9], conventional generation control was combined with ED automatically and dynamically, providing optimization views to redesign frequency control. In [10], [11], optimal load control schemes were proposed, which were used to balance power and/or restore the nominal frequency after a disturbance.

In this paper, we merge and redesign conventional primary and secondary frequency control for power systems. Moreover, we consider controllable loads [12] participating in active power regulation. Unlike most previous work, we propose a saddle point design framework that enables power networks to solve a general DC Optimal Power Flow (OPF) problem dynamically, i.e., to achieve Real-time Economic Dispatch (RED). The key steps of our approach include: (i) reverse-engineer the power system model as a primal-dual gradient dynamics [13], [9] to solve a saddle point problem; (ii) derive a second saddle point problem for an ED problem which is formulated as a DC OPF problem where the power network congestion constraints are considered; (iii) combine the two saddle point problems and derive a distributed dynamic feedback controller through optimization decomposition methods. The resulting controller is completely distributed and scalable, only requiring each bus communicating with its neighbors. Moreover, the overall system asymptotically converges to an equilibrium point where the DC OPF problem is solved and the power network congestion constraints are satisfied. This saddle point design approach is powerful and general in the sense that it allows us to design control rules that drive the system to a steady state solving other convex optimization problems if we replace the DC OPF problem with the corresponding optimization problem.

The rest of the paper is organized as follows: In Section II, we present the model of a transmission level network under exogenous disturbances, and formulate the RED problem. In Section III, we propose a saddle point design framework to solve the RED problem. In Section IV, we present numerical examples using the IEEE 14-bus network to illustrate the properties of the resulting control scheme. Conclusions and future work are presented in Section V.

II. Problem Setup

A. Power Network Model

Consider a transmission level network with arbitrary topology, described by a connected undirected graph $(\mathcal{G} \bigcup \mathcal{L}, \mathcal{E})$. Here $\mathcal{G}$ is the set of generator buses, $\mathcal{L}$ is the set of load

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buses, and $\mathcal{E} \subseteq (\mathcal{G} \bigcup \mathcal{L}) \times (\mathcal{G} \bigcup \mathcal{L})$ is the set of transmission lines. Each generator bus contains only one synchronous generator, and each load bus contains only one load which is an aggregation of a certain number of users at the bus it is connected to (alternatively speaking, each load corresponds to a distribution level network). This can be realized through introducing fictitious buses [14]. View all buses as voltage sources and let the voltage of each bus be $v_i \angle \delta_i, i \in \mathcal{G} \bigcup \mathcal{L}$, where δ_i is the voltage phase angle with respect to the rotating speed ω_i^0, e.g., $2\pi \times 50$Hz in Europe. Assume that the network is working around a nominal operating point which is determined by an ED problem at a more slower time-scale. In this paper, we make the following assumption, which is commonly used in transmission level networks [15], [1].

Assumption 1: All bus voltage magnitudes are fixed, i.e., $v_i, i \in \mathcal{G} \bigcup \mathcal{L}$ are constant. The resistance of transmission lines is negligible. Reactive power injections and flows are omitted.

To simplify the notation, in the rest of the paper, all state variables denote *time-varying deviations* from their nominal operating values. Let $\omega_i = \dot{\delta}_i, i \in \mathcal{G} \bigcup \mathcal{L}$ and $\delta_{ij} = -\delta_{ji} = \delta_i - \delta_j, (i,j) \in \mathcal{E}$. The dynamic model we use for the network is derived through linearization and is given by [14], [1], [6]

$$M_i \dot{\omega}_i + D_i \omega_i = P_{M_i} - d_i - \sum_{(i,j)\in\mathcal{E}} T_{ij} \delta_{ij}, \ i \in \mathcal{G} \tag{1a}$$

$$D_i \omega_i = -P_{L_i} - d_i - \sum_{(i,j)\in\mathcal{E}} T_{ij} \delta_{ij}, \ i \in \mathcal{L} \tag{1b}$$

$$\dot{\delta}_{ij} = \omega_i - \omega_j, \ (i,j) \in \mathcal{E} \tag{1c}$$

$$\dot{P}_{M_i} = T_{TG_i}^{-1} (P_{C_i} - P_{M_i} - R_i^{-1} \omega_i), \ i \in \mathcal{G}. \tag{1d}$$

Here $M_i > 0$ is the generator inertia, $D_i > 0$ is the damping coefficient, P_{M_i} is the mechanical power input, P_{L_i} is the power consumption of controllable loads, d_i is the disturbance injection, e.g., variations on both supply and demand, and renewable energy injections. Also, $T_{ij} = v_i^0 v_j^0 B_{ij} \cos(\delta_i^0 - \delta_j^0), (i,j) \in \mathcal{E}$, where variables with the subscript zero denote their nominal operating values, and $B_{ij} = B_{ji} > 0$ is the susceptance of the transmission line connecting buses i and j. The turbine-governor system for each generator is modeled by Equation (1d), where P_{C_i} is the command/control input to the generator and $T_{TG_i}, R_i > 0$ are constant parameters. Note that we have simplified the dynamics of the turbine-governor system using a first-order model as in [6].

Remark 1: Given $P_{C_i}, d_i, i \in \mathcal{G}$ and $P_{L_i}, d_i, i \in \mathcal{L}$, the equilibrium point of system (1) is:

$$\omega_i^* = \omega^* = \frac{\sum_{i\in\mathcal{G}} (P_{C_i} - d_i) - \sum_{i\in\mathcal{L}} (P_{L_i} + d_i)}{\sum_{i\in\mathcal{G}\bigcup\mathcal{L}} D_i + \sum_{i\in\mathcal{G}} R_i^{-1}}, i \in \mathcal{G} \tag{2a}$$

$$\delta_{ij}^* = \delta_i^* - \delta_j^*, \ (i,j) \in \mathcal{E} \tag{2b}$$

$$P_{M_i}^* = P_{C_i} - R_i^{-1} \omega_i^*, \ i \in \mathcal{G} \tag{2c}$$

where $\delta_i^*, i \in \mathcal{G} \bigcup \mathcal{L}$ is the solution of

$$\sum_{(i,j)\in\mathcal{E}} T_{ij} (\delta_i - \delta_j) = P_{C_i} - d_i - (D_i + R_i^{-1}) \omega_i^*, \ i \in \mathcal{G}$$

$$\sum_{(i,j)\in\mathcal{E}} T_{ij} (\delta_i - \delta_j) = -P_{L_i} - d_i - D_i \omega^*, \ i \in \mathcal{L}.$$

The above equations result in multiple solutions for $\delta_i, i \in \mathcal{G} \bigcup \mathcal{L}$ but unique $\delta_{ij} = \delta_i - \delta_j, (i,j) \in \mathcal{E}$. Thus, we conclude that given $P_{C_i}, d_i, i \in \mathcal{G}$ and $P_{L_i}, d_i, i \in \mathcal{L}$, the states ω_i, δ_{ij} and P_{M_i} of system (1) will converge to a *unique* equilibrium point of system (1) as above (the convergence will be shown later). This is in accord with Remark 5 in [10]. □

B. Real-time Economic Dispatch (RED)

Suppose that the network is operating around a nominal state determined by an ED problem at a more slower time-scale, and that a constant disturbance $d_i, i \in \mathcal{G} \bigcup \mathcal{L}$ occurs. As a result, the frequency deviates from the nominal value. It is expected that generators and controllable loads can adjust their power generation $P_{C_i}, i \in \mathcal{G}$ and consumption $P_{L_i}, i \in \mathcal{L}$ respectively in real-time using *local* information, not only to release the overall system from stress, i.e., to restore the grid frequency, but also to result in a better supply-demand balance, e.g., minimizing generation cost, maximizing load utility, and satisfying transmission line congestion constraints. These goals lead to a steady-state optimization/DC OPF problem:

$$\min_{P_{M_i}, P_{L_i}, \delta_i} \sum_{i\in\mathcal{G}} C_i(P_{M_i}) - \sum_{i\in\mathcal{L}} U_i(P_{L_i}) \tag{3a}$$

$$\text{subject to } P_{M_i} - d_i - \sum_{(i,j)\in\mathcal{E}} T_{ij}(\delta_i - \delta_j) = 0, \ i \in \mathcal{G} \tag{3b}$$

$$- P_{L_i} - d_i - \sum_{(i,j)\in\mathcal{E}} T_{ij}(\delta_i - \delta_j) = 0, \ i \in \mathcal{L} \tag{3c}$$

$$P_{C_i}^{\min} \le P_{M_i} \le P_{C_i}^{\max}, \ i \in \mathcal{G} \tag{3d}$$

$$P_{L_i}^{\min} \le P_{L_i} \le P_{L_i}^{\max}, \ i \in \mathcal{L} \tag{3e}$$

$$P_{TC_{ij}}^{\min} \le T_{ij}(\delta_i - \delta_j) \le P_{TC_{ij}}^{\max}, \ (i,j) \in \mathcal{E} \tag{3f}$$

where $C_i(P_{M_i})$ is the cost function for each generator, $U_i(P_{L_i})$ is the Utility function for each controllable load, Equations (3b)-(3c) represent power flow balance at each bus, Equations (3d)-(3f) are capacity constraints for generators, loads, and transmission lines respectively, and $P_{C_i}^{\min}, P_{C_i}^{\max}, i \in \mathcal{G}, P_{L_i}^{\min}, P_{L_i}^{\max}, i \in \mathcal{L}, P_{TC_{ij}}^{\min}, P_{TC_{ij}}^{\max}, (i,j) \in \mathcal{E}$ are corresponding capacity bounds. Note that by setting $P_{TC_{ij}}^{\max} = P_{TC_{ij}}^{\min}$, the scheduled power flow in line (i,j) can be maintained. For (3), we have a related assumption, which is extensively used in the literature [6], [10], [3], [4].

Assumption 2: Problem (3) is feasible. Moreover, each C_i is a strictly convex function in P_{M_i} and each U_i is a strictly concave function in P_{L_i}.

Since the dynamics of P_{M_i} and δ_i are constrained by (1) during the transient, they therefore cannot be instantaneously set to the solution of problem (3). Note that Equations (3b)-(3c) require the frequency to be restored when system (1) reaches steady state, i.e., $\omega_i^* = 0, i \in \mathcal{G} \bigcup \mathcal{L}$. According to Remark 1, this results in $P_{M_i}^* = P_{C_i}, i \in \mathcal{G}$ in steady state. So we can reformulate problem (3) as one that can be used to design $P_{C_i}, i \in \mathcal{G}$ and $P_{L_i}, i \in \mathcal{L}$, given by

$$\min_{P_{C_i}, P_{L_i}, \theta_i} \sum_{i\in\mathcal{G}} C_i(P_{C_i}) - \sum_{i\in\mathcal{L}} U_i(P_{L_i}) \tag{4a}$$

$$\text{subject to } P_{C_i} - d_i - \sum_{(i,j)\in\mathcal{E}} T_{ij}(\theta_i - \theta_j) = 0, \ i \in \mathcal{G} \tag{4b}$$

$$-P_{L_i}-d_i-\sum_{(i,j)\in\mathcal{E}}T_{ij}(\theta_i-\theta_j)=0,\ i\in\mathcal{L} \quad (4c)$$

$$P_{C_i}^{\min}\le P_{C_i}\le P_{C_i}^{\max},\ i\in\mathcal{G} \quad (4d)$$

$$P_{L_i}^{\min}\le P_{L_i}\le P_{L_i}^{\max},\ i\in\mathcal{L} \quad (4e)$$

$$P_{TC_{ij}}^{\min}\le T_{ij}(\theta_i-\theta_j)\le P_{TC_{ij}}^{\max},\ (i,j)\in\mathcal{E} \quad (4f)$$

where $\theta_i, i\in\mathcal{G}\bigcup\mathcal{L}$ are ancillary decision variables. In fact, we have replaced P_{M_i},δ_i in (3) by P_{C_i},θ_i to derive (4).

Proposition 1: Given the control input $P_{C_i}^*, i\in\mathcal{G}$ and $P_{L_i}^*, i\in\mathcal{L}$ to system (1), where $(P_{C_i}^*,P_{L_i}^*,\theta_i^*)$ is the optimal solution of (4), the equilibrium point of system (1) satisfies $P_{M_i}^*=P_{C_i}^*, i\in\mathcal{G}$ and $\delta_{ij}^*=\theta_i^*-\theta_j^*, (i,j)\in\mathcal{E}$. As a result, $(P_{M_i}^*,P_{L_i}^*,\delta_i^*)$ is an optimal solution of problem (3).

According to the above proposition, the optimality of problem (3) is preserved after the reformulation. Finally, the Real-time Economic Dispatch problem is described as: design $P_{C_i}, i\in\mathcal{G}$ and $P_{L_i}, i\in\mathcal{L}$ so that for given $d_i, i\in\mathcal{G}\bigcup\mathcal{L}$, system (1) is driven to an equilibrium point where the steady-state optimization problem (3)/(4) is solved.

III. A Saddle Point Design Framework

A. Reverse-engineering

In this subsection, we reverse-engineer system (1) as a primal-dual gradient dynamics [13], [9] to solve a saddle point problem. The resulting saddle point problem will allow us to embed the steady-state optimization problem for control design. We first present two lemmas relating to properties of a saddle point of a function and primal-dual gradient dynamics.

Lemma 1: Let $f(x,y)\in\mathcal{C}^2$: $\mathbb{R}^a\times\mathbb{R}^b\to\mathbb{R}$ satisfy: for all x,y, $\nabla_x^2 f\succeq 0, \nabla_y^2 f\preceq 0$. Then $(\tilde{x},\tilde{y})$ is a saddle point of f, i.e., $f(\tilde{x},y)\le f(\tilde{x},\tilde{y})\le f(x,\tilde{y})$, if and only if $\nabla_{x,y}f|_{x=\tilde{x},y=\tilde{y}}=\mathbf{0}$. Here $\mathcal{C}^2$ is the class of functions that are 2 times continuously differentiable. $\nabla_x^2 f(x)$ denotes the Hessian matrix of $f(x)$ with respect to x. $X\succeq 0$ ($X\succ 0$) denotes that a square matrix X is positive semi-definite (positive definite). $\nabla_x f(x)$ denotes the gradient (as a row vector) of a scalar function $f(x)$ with respect to x.

Proof: This Lemma is a corollary of the Karush-Kuhn-Tucker (KKT) conditions, the proof of which can be found in [16]. ■

Lemma 2: Let $f(x,y)\in\mathcal{C}^2$: $\mathbb{R}^a\times\mathbb{R}^b\to\mathbb{R}$ satisfy: for all x,y, $\nabla_x^2 f\succeq 0, \nabla_y^2 f\preceq 0$. Then the trajectories of the primal-dual dynamics [13]/saddle point dynamics [9] given by

$$\dot{x}=-K_x\left(\frac{\partial f}{\partial x}\right)^T \quad (5a)$$

$$\dot{y}=K_y\left(\frac{\partial f}{\partial y}\right)^T \quad (5b)$$

are bounded, where $K_x\in\mathbb{R}^{a\times a}, K_y\in\mathbb{R}^{b\times b}$ are positive definite constant matrices. Furthermore, if either $\nabla_x^2 f\succ 0$ or $\nabla_y^2 f\prec 0$ holds, the trajectories of (5) asymptotically converge to a saddle point of f.

Proof: The proof follows the content of Section 2 in [13] and Proposition 13 in [9]. ■

Now consider a candidate Lagrangian $L_0(\delta_{ij},P_{M_i},\omega_i)\in\mathcal{C}^2:\mathbb{R}^{|\mathcal{E}|}\times\mathbb{R}^{|\mathcal{G}|}\times\mathbb{R}^{|\mathcal{G}|}\to\mathbb{R}$ given by

$$\begin{aligned}L_0=&-\sum_{i\in\mathcal{G}}\frac{1}{2D_i}\left(D_i\omega_i-P_{M_i}+d_i+\sum_{(i,j)\in\mathcal{E}}T_{ij}\delta_{ij}\right)^2\\&+\sum_{i\in\mathcal{G}}\frac{1}{2D_i}\left(P_{M_i}-d_i-\sum_{(i,j)\in\mathcal{E}}T_{ij}\delta_{ij}\right)^2\\&+\sum_{i\in\mathcal{L}}\frac{1}{2D_i}\left(P_{L_i}+d_i+\sum_{(i,j)\in\mathcal{E}}T_{ij}\delta_{ij}\right)^2\\&+\sum_{i\in\mathcal{G}}\frac{R_i}{2}(P_{M_i}-P_{C_i})^2\end{aligned} \quad (6)$$

where $\delta_{ij},P_{M_i},\omega_i$ are decision variables[1], and P_{C_i},P_{L_i},d_i are constant. Note that for simplicity, we have used the same notation as in the dynamics (1). Since $\nabla_{\delta_{ij},P_{M_i}}^2 L_0\succeq 0, \nabla_{\omega_i}^2 L_0\prec 0$ are true, L_0 is convex in δ_{ij},P_{M_i} and strictly concave in ω_i. We can then formulate a saddle point problem as

$$\min_{\delta_{ij},P_{M_i}}\max_{\omega_i} L_0. \quad (7)$$

Through straightforward derivation, we can show that system dynamics (1) is actually the saddle point dynamics (5) of L_0, due to the following equalities:

$$\frac{\partial L_0}{\partial\omega_i}=M_i\dot{\omega}_i,\ i\in\mathcal{G}\ (\dot{\omega}_i \text{ is given in (1a)})$$

$$\frac{\partial L_0}{\partial\delta_{ij}}=-T_{ij}\dot{\delta}_{ij},\ (i,j)\in\mathcal{E}\ (\dot{\delta}_{ij} \text{ is given in (1c)})$$

$$\frac{\partial L_0}{\partial P_{M_i}}=-T_{TG_i}R_i\dot{P}_{M_i},\ i\in\mathcal{G}\ (\dot{P}_{M_i} \text{ is given in (1d)}).$$

Because of Lemmas 1-2, the following theorem can be proven.

Theorem 1: The trajectories of system (1) asymptotically converge to a saddle point of problem (7).

B. A Saddle Point Design Approach

We now propose a saddle point design approach to solve the RED problem defined in Section II-B. Formulate the Lagrangian of (4) as

$$\begin{aligned}&L_p(P_{C_i},P_{L_i},\theta_i,\zeta_i,\lambda_i,\mu_i^+,\mu_i^-,\nu_i^+,\nu_i^-,l_{ij}^+,l_{ij}^-)=\sum_{i\in\mathcal{G}}C_i(P_{C_i})\\&-\sum_{i\in\mathcal{L}}U_i(P_{L_i})+\sum_{i\in\mathcal{G}}\zeta_i\left(P_{C_i}-d_i-\sum_{(i,j)\in\mathcal{E}}T_{ij}(\theta_i-\theta_j)\right)\\&+\sum_{i\in\mathcal{L}}\lambda_i\left(-P_{L_i}-d_i-\sum_{(i,j)\in\mathcal{E}}T_{ij}(\theta_i-\theta_j)\right)+\sum_{i\in\mathcal{G}}\mu_i^+(P_{C_i}\\&-P_{C_i}^{\max})+\sum_{i\in\mathcal{G}}\mu_i^-(P_{C_i}^{\min}-P_{C_i})+\sum_{i\in\mathcal{L}}\nu_i^+(P_{L_i}-P_{L_i}^{\max})\\&+\sum_{i\in\mathcal{L}}\nu_i^-(P_{L_i}^{\min}-P_{L_i})+\sum_{(i,j)\in\mathcal{E}}l_{ij}^+(T_{ij}(\theta_i-\theta_j)-P_{TC_{ij}}^{\max})\\&+\sum_{(i,j)\in\mathcal{E}}l_{ij}^-(P_{TC_{ij}}^{\min}-T_{ij}(\theta_i-\theta_j))\end{aligned} \quad (8)$$

where $\zeta_i,\mu_i^+,\mu_i^-, i\in\mathcal{G}$, $\lambda_i,\nu_i^+,\nu_i^-, i\in\mathcal{L}$ and $l_{ij}^+,l_{ij}^-,(i,j)\in\mathcal{E}$ are Lagrange multipliers (dual variables) for the constraints in (4). For convenience, denote $\alpha=(\mu_i^+,\mu_i^-,\nu_i^+,\nu_i^-,l_{ij}^+,l_{ij}^-)$

[1]Here we require $\delta_{ij}=-\delta_{ji},(i,j)\in\mathcal{E}$. Thus there is only one free decision variable between δ_{ij} and δ_{ji}.

and $\alpha \geq 0$ means $\mu_i^+, \mu_i^-, \nu_i^+, \nu_i^-, l_{ij}^+, l_{ij}^- \geq 0$. We then obtain the following saddle point problem:

$$\min_{P_{C_i}, P_{L_i}, \theta_i} \max_{\alpha \geq 0, \zeta_i, \lambda_i} L_p. \tag{9}$$

Under Assumption 2, strong duality holds [16]. As a result, solving problem (4) is equivalent to solving problem (9). Consider the augmented saddle point problem given by

$$\min_{\delta_{ij}, P_{M_i}, P_{C_i}, P_{L_i}, \theta_i} \max_{\alpha \geq 0, \zeta_i, \lambda_i, \omega_i} L_{red} = L_0 + \gamma L_p \tag{10}$$

where $\gamma > 0$ is constant. We then have a related proposition regarding the saddle point problems (7), (9) and (10).

Proposition 2: If $(\omega_i^*, \delta_{ij}^*, P_{M_i}^*, P_{C_i}^*, P_{L_i}^*, \theta_i^*, \zeta_i^*, \lambda_i^*, \alpha^*)$ is a saddle point of L_{red}, $(\omega_i^*, \delta_{ij}^*, P_{M_i}^*)$ is a saddle point of L_0 and $(P_{C_i}^*, P_{L_i}^*, \theta_i^*, \zeta_i^*, \lambda_i^*, \alpha^*)$ is a saddle point of L_p.

Since $\nabla^2_{\delta_{ij}, P_{M_i}, P_{C_i}, P_{L_i}, \theta_i} L_{red} \succeq 0$ and $\nabla^2_{\omega_i, \zeta_i, \lambda_i, \alpha} L_{red} \preceq 0$ hold, under Lemma 2, the trajectories of the primal-dual gradient dynamics given by Equation (11) are bounded. Note that the projection in (11j)-(11o) does not affect the result in Lemma 2 [13]. In Equation (11), $\{\zeta_i, \lambda_i\} = \zeta_i$ when $i \in \mathcal{G}$ and $\{\zeta_i, \lambda_i\} = \lambda_i$ when $i \in \mathcal{L}$, $K_{C_i}, K_{\theta_i}, K_{\zeta_i}, K_{\mu_i^+}, K_{\mu_i^-}, i \in \mathcal{G}$, $K_{L_i}, K_{\theta_i}, K_{\lambda_i}, K_{\nu_i^+}, K_{\nu_i^-}, i \in \mathcal{L}$, $K_{l_{ij}^+}, K_{l_{ij}^-}, (i,j) \in \mathcal{E}$ are all positive constant representing the controller gains, $l_{ij}^+ = l_{ji}^-, (i,j) \in \mathcal{E}$ is always true, and $(h(y))_x^+$ is the positive projection of a function $h(y)$ on a variable x given by

$$(h(y))_x^+ = \begin{cases} h(y) & \text{if } x > 0 \\ \max(0, h(y)) & \text{if } x = 0 \end{cases}.$$

Power network dynamics:

$$M_i \dot{\omega}_i + D_i \omega_i = P_{M_i} - d_i - \sum_{(i,j)\in\mathcal{E}} T_{ij}\delta_{ij}, \ i \in \mathcal{G} \tag{11a}$$

$$D_i \omega_i = -P_{L_i} - d_i - \sum_{(i,j)\in\mathcal{E}} T_{ij}\delta_{ij}, \ i \in \mathcal{L} \tag{11b}$$

$$\dot{\delta}_{ij} = \omega_i - \omega_j, \ (i,j) \in \mathcal{E} \tag{11c}$$

$$\dot{P}_{M_i} = T_{TG_i}^{-1}(P_{C_i} - P_{M_i} - R_i^{-1}\omega_i), \ i \in \mathcal{G} \tag{11d}$$

Control input dynamics:

$$\dot{P}_{C_i} = K_{C_i}(R_i(P_{M_i} - P_{C_i}) - \gamma(C_i'(P_{C_i}) + \zeta_i + \mu_i^+ - \mu_i^-)) \quad i \in \mathcal{G} \tag{11e}$$

$$\dot{P}_{L_i} = K_{L_i}(\omega_i + \gamma(U_i'(P_{L_i}) + \lambda_i - \nu_i^+ + \nu_i^-)), \ i \in \mathcal{L} \tag{11f}$$

Ancillary variable dynamics:

$$\dot{\theta}_i = K_{\theta_i} \sum_{(i,j)\in\mathcal{E}} T_{ij}(\{\zeta_i, \lambda_i\} - \{\zeta_j, \lambda_j\} - l_{ij}^+ + l_{ij}^-), \ i \in \mathcal{G} \bigcup \mathcal{L} \tag{11g}$$

$$\dot{\zeta}_i = K_{\zeta_i}\left(P_{C_i} - d_i - \sum_{(i,j)\in\mathcal{E}} T_{ij}(\theta_i - \theta_j)\right), \ i \in \mathcal{G} \tag{11h}$$

$$\dot{\lambda}_i = K_{\lambda_i}\left(-P_{L_i} - d_i - \sum_{(i,j)\in\mathcal{E}} T_{ij}(\theta_i - \theta_j)\right), \ i \in \mathcal{L} \tag{11i}$$

$$\dot{\mu}_i^+ = K_{\mu_i^+}(P_{C_i} - P_{C_i}^{\max})^+_{\mu_i^+}, \ i \in \mathcal{G} \tag{11j}$$

$$\dot{\mu}_i^- = K_{\mu_i^-}(P_{C_i}^{\min} - P_{C_i})^+_{\mu_i^-}, \ i \in \mathcal{G} \tag{11k}$$

$$\dot{\nu}_i^+ = K_{\nu_i^+}(P_{L_i} - P_{L_i}^{\max})^+_{\nu_i^+}, \ i \in \mathcal{L} \tag{11l}$$

$$\dot{\nu}_i^- = K_{\nu_i^-}(P_{L_i}^{\min} - P_{L_i})^+_{\nu_i^-}, \ i \in \mathcal{L} \tag{11m}$$

$$\dot{l}_{ij}^+ = K_{l_{ij}^+}(T_{ij}(\theta_i - \theta_j) - P_{TC_{ij}}^{\max})^+_{l_{ij}^+}, \ (i,j) \in \mathcal{E} \tag{11n}$$

$$\dot{l}_{ij}^- = K_{l_{ij}^-}(P_{TC_{ij}}^{\min} - T_{ij}(\theta_i - \theta_j))^+_{l_{ij}^-}, \ (i,j) \in \mathcal{E} \tag{11o}$$

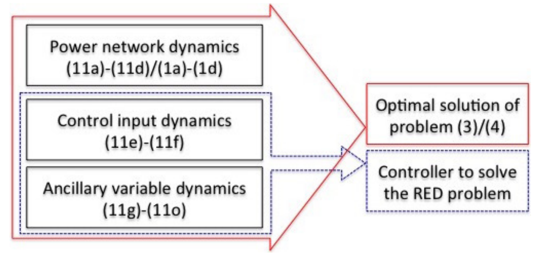

Fig. 1: Block structure for system (11).

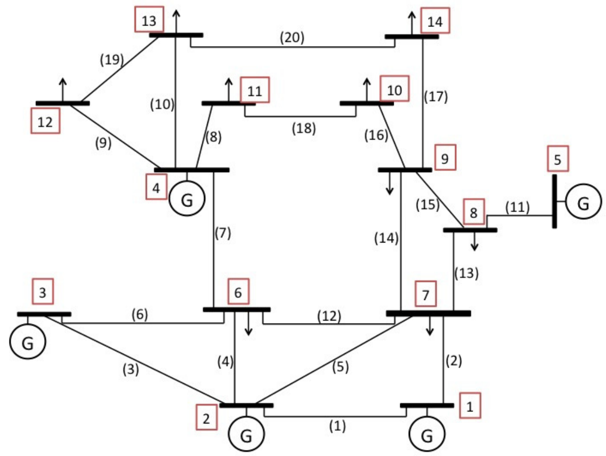

Fig. 2: IEEE 14-bus network, with 5 generator buses, 9 load buses and 20 transmission lines. The buses are numbered with red blocks. The transmission lines are numbered with brackets.

It is important to note that controller (11e)-(11o) is completely *distributed*, i.e., states are updated using only local information and signals from their neighborhood.

Theorem 2: Under Assumption 2, the trajectories of system (11) asymptotically converge to a saddle point of L_{red}, denoted by $(\omega_i^*, \delta_{ij}^*, P_{M_i}^*, P_{C_i}^*, P_{L_i}^*, \theta_i^*, \zeta_i^*, \lambda_i^*, \alpha^*)$. Moreover, $(P_{M_i}^*, P_{L_i}^*, \delta_{ij}^*)$ is optimal solution of problem (3).

Theorem 2 immediately leads to the optimality and stability of system (11), whose block structure is show in Figure 1. The benefit of the saddle point design approach is summarized as follows. First, the approach allows us to embed different kinds of steady-state convex optimization problems, where problem (4) is a typical example. Second, as long as the optimization problem is distributed, e.g, with separable objective functions and local constraints, the resulting controller is completely distributed. Third, the trajectories of the closed-loop system are bounded, and can be further proven to converge to an equilibrium point where the optimization problem is solved, as illustrated in Theorem 2.

IV. Numerical Investigations

We now present numerical investigations using the IEEE 14-bus network illustrated in Figure 2. The parameters of the

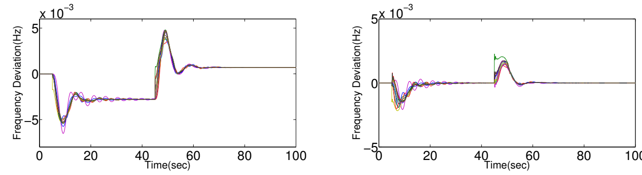

Fig. 3: Bus frequency deviations. Left: uncontrolled case. Right: controlled case.

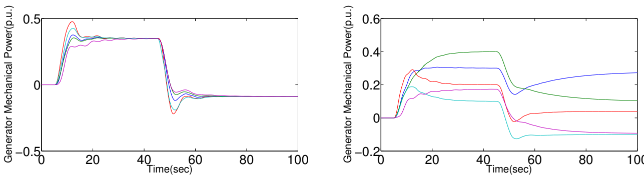

Fig. 4: Generator mechanical power. Left: uncontrolled case. Right: controlled case.

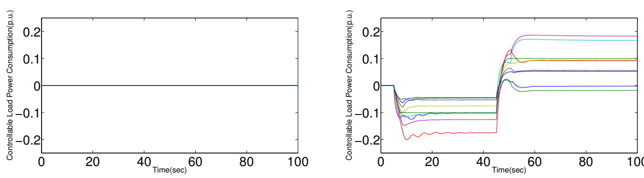

Fig. 5: Controllable load power consumption. Left: uncontrolled case. Right: controlled case.

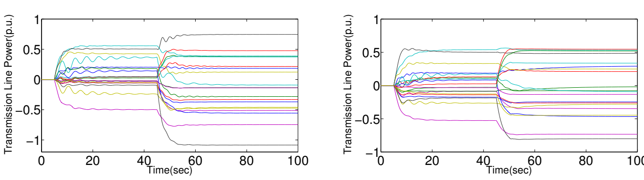

Fig. 6: Transmission line power flow. Left: uncontrolled case. Right: controlled case.

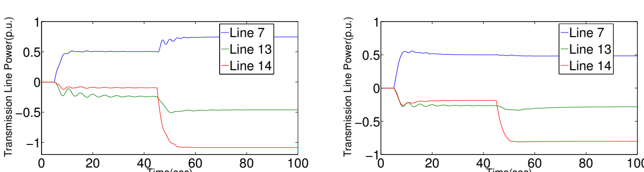

Fig. 7: Power flow through line 7 (blue), 13 (green), 14 (red). Left: uncontrolled case. Right: controlled case.

overall system are randomly selected within ranges given by: $M_i \in [8,12]$, $D_i \in [0.3,2]$, $T_{ij} \in [20,30]$, $T_{TG_i} \in [3,7]$, $R_i = 0.05$, $\gamma = 2$, $K_{C_i} = K_{L_i} = 15$, $K_{\theta_i} = K_{\zeta_i} = K_{\lambda_i} = K_{\mu_i^+} = K_{\mu_i^-} = K_{\nu_i^+} = K_{\nu_i^-} = K_{l_{ij}^+} = K_{l_{ij}^-} = 10$, $C_i(P_{C_i}) = \frac{c_i}{2}P_{C_i}^2$ and $U_i(P_{L_i}) = -\frac{c_i}{2}P_{L_i}^2$ where $c_i \in [0.01, 0.2]$, $P_{C_i}^{\max} = -P_{C_i}^{\min} \in [0.1, 0.5]$, $P_{L_i}^{\max} = -P_{L_i}^{\min} \in [0.1, 0.5]$, $P_{TC_{ij}}^{\max} = -P_{TC_{ij}}^{\min} = 1$ except $P_{TC_{46}}^{\max} = -P_{TC_{46}}^{\min} = 0.5$ (corresponding to line 7), $P_{TC_{78}}^{\max} = -P_{TC_{78}}^{\min} = 0.3$ (corresponding to line 13) and $P_{TC_{79}}^{\max} = -P_{TC_{79}}^{\min} = 0.8$ (corresponding to line 14), all in per unit (p.u.).

We consider a scenario consisting of two disturbances which is realized as follows: the system is stabilized at the nominal operating point at $t = 0$s; at $t = 5$s, there is a step change $d_i = +2$p.u. at bus 6; 40s later, there is a step change $d_i = -2.5$p.u. at bus 9. The simulation results are shown in Figures 3-7. Compared with the uncontrolled case (shown in figures on the left), we can see that the proposed control scheme restores the grid frequency quickly under disturbances. Moreover, it can be verified that the optimization problem is solved when system reaches steady state, i.e., generation cost is minimized within capacity limit, load utility is maximized within capacity limit, and transmission line congestion constraints are satisfied.

V. Conclusion and Future Work

In this paper, we have proposed a saddle point design framework to solve the Real-time Economic Dispatch problem. Under exogenous disturbances, the distributed controller derived through this framework asymptotically stabilized the power network at an equilibrium point which solved the steady-state optimization problem (3)/(4).

Future work will focus on nonlinear power system dynamics under this framework. Another direction is to apply the framework to solve tracking problems for linear systems.

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